Criticality of Large Delay Tolerant Networks via Directed Continuum Percolation in Space-Time

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***Abstract*—We study delay tolerant networking (DTN) and in particular, its capacity to store, carry and forward messages to their final destination(s). We approach this broad question in the framework of percolation theory. To this end, we assume an elementary mobility model, where nodes arrive to an infinite plane according to a Poisson point process, move a certain distance ℓ, and then depart. In this setting, we characterize the mean density of nodes required to support DTN style networking. Under the given assumptions, we show that DTN communication is feasible when the mean node degree ν is greater than $4 \cdot \eta_c(\gamma)$, where parameter $\gamma = \ell/\mathrm{d}$ is the ratio of the distance ℓ to the transmission range d, and $\eta_c(\gamma)$ is the critical reduced number density of tilted cylinders in a directed continuum percolation model. By means of Monte Carlo simulations, we give numerical values for $\eta_c(\gamma)$. The asymptotic behavior of $\eta_c(\gamma)$ when γ tends to ∞ is also derived from a fluid flow analysis.**

***Index Terms*—DTN, capacity, percolation, criticality, mobility**

I. Introduction

Delay-tolerant networking (DTN) is a last-resort networking paradigm for mobile nodes when direct and multi-hop connections are infeasible, i.e., the network is not connected and nodes have to *carry* the messages to the next node (store-carry-forward). We study the capacity of DTN to deliver a message to its destination(s) in the framework of the percolation theory. To this end, we assume an elementary mobility model where nodes arrive according to a Poisson point process on a plane, move a certain distance ℓ, and then depart. Nodes perform epidemic routing, i.e., whenever two nodes meet, all messages are exchanged [6]–[8]. We are interested in finding the circumstances under which the lifetime of a message becomes infinite (with some positive probability) so that the message could reach a recipient located at an arbitrary distance from the source. We obtain a fundamental criticality condition that characterizes the sufficient mean density of nodes to this end. The criticality condition takes form $\nu > \nu_c(\gamma) = 4\,\eta_c(\gamma)$, where ν denotes the *mean node degree* (number of neighbors), γ is the ratio of ℓ to the transmission range d, and $\eta_c(\gamma)$ is an unknown function, which we determine in this paper.

For a mobile DTN, we find that $\nu_c(\gamma) \leq 1.52$ for all γ. In contrast, (non-DTN) multi-hop ad-hoc communication becomes feasible only at $\nu \approx 4.51$, where a gigantic connected component emerges [1]. In other words, DTN communication is possible over a very sparse network provided that the network's topology changes in time. A convenient characterization in fact is to say that *DTN is a network that is super-critical in space-time.* In practice, DTN is often sub-critical at a random time instant and the messages reach the destinations through the store-carry-forward routing in space-time.

This work is motivated by different opportunistic networking schemes. One such scheme is *Floating Content*, where nodes replicate messages only within the area where each message is deemed relevant. A criticality condition for long mean message lifetime was established in [2] assuming mobile nodes and point contacts (a short transmission range compared to the dimension of the area). Similarly, in Beachnet [3] the aim is to produce quasi-periodic "information waves" carried by an underlying field of immobile nodes (say, devices on a beach). Similar concepts for one-to-many content dissemination include hovering information [4] and ad-hoc podcasting [5], but our model naturally covers the special case of one-to-one messaging. DTN-style communication introduces many interesting problems. Perhaps the most fundamental question is whether a network can transport messages to their intended destinations or not, and this is also our focus in this paper.

Percolation theory [9], [10] has been succesfully applied to study the performance of wireless multi-hop networks since the early work by Gilbert [11]. Similarly as in [11], most of the work utilizes undirected planar continuum percolation models to argue, e.g., about the network's connectivity or capacity. In contrast, in [12], we assumed *stationary nodes* and showed that opportunistic content dissemination schemes can be analyzed by using a three-dimensional continuum percolation model. The third dimension is time and, due to the causality, only the post-contact part of an object belongs to the given cluster. We adapt the same approach, but instead of approximating the process by undirected percolation model, we consider the *actual directed percolation* characterizing the dissemination of messages by *mobile nodes* exactly in a DTN network. In this setting, we obtain a fundamental result for the minimum mean node degree for DTN communication as a function of the ratio of movement to the transmission range.

II. Model and Notations

Let us start by describing the model, including the arrival process, nodes' mobility and the assumptions regarding the radio communication. Throughout this paper, we implicitly assume a large network and a long distance, either in space and/or in time, between the source and the destination.

1) Node mobility: We assume that nodes arrive according to a Poisson point process on an infinite plane with rate density λ [node/m^2/s]. Mobility is specified by distance ℓ and

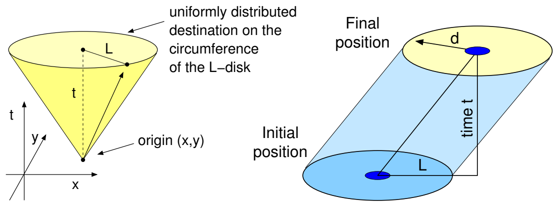


Figure 1: *Node mobility in space-time: (left) during time t, a node moves a distance of $\ell = L$ to a random direction in (x, y)-plane and departs. (right) The volume covered corresponds to a tilted cylinder.*

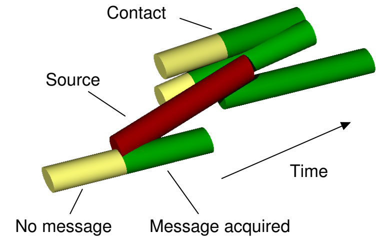


Figure 2: *Small sample realization with γ=1 and η=0.3.*

time duration t: during time t nodes move distance ℓ to a random direction (isotropic), and depart. Both ℓ and t are fixed. The arrival-departure cycle can be interpreted as the time when a mobile node is participating in the DTN activity, e.g., the "departure" can correspond to an information content replacement event in the node. The node density in plane is

$$\tilde{n} \triangleq \lambda \cdot t,$$

In the basic case, $\ell = 0$ and the nodes remain stationary for the time t and depart. The node mobility model applies to the intermediate nodes, while the source and destination node can be two permanent nodes located far apart from each other.

2) Information dissemination: Whenever two nodes are within each others' transmission range d, they immediately exchange the messages. The fixed transmission range is often referred to as the Gilbert's disc or boolean model [11]. The mean number of neighbors a node has, i.e., the mean node degree, is $\nu = \tilde{n} \cdot \pi d^2$. In the percolation theory, two objects are connected if they overlap and thus the corresponding radius r is half of the transmission range,

$$r = d/2.$$

3) Space-time: The movement and the radio coverage of a node in space-time, comprising (x, y)-plane and the time axis, are depicted in Fig. 1. The radio coverage at any time instant is a disc. As nodes move in time, the coverage in 3D *space-time* is a sheared cylinder. The *tilt* of the cylinder is defined by the ratio of the movement to the transmission range,

$$\gamma \triangleq \frac{\ell}{d} = \frac{\ell}{2r}.$$

Somewhat counter-intuitively, the number density of nodes (cylinders) in *space-time* is equal to the arrival rate λ,[1]

$$n = \lambda.$$

Instead of number density, the density of objects is often expressed using either the *reduced number density* η,

$$\eta \triangleq n \cdot V,$$

where V is the volume of the shape, or the *volume fraction* ϕ, for which it holds that $\phi = 1 - e^{-\eta}$.

In our case, $V = \pi r^2 \cdot t$ independently of γ. Without lack of generality, we can scale the space-time. The time axis can be scaled by constant α, e.g., so that $t = 1$. Similarly, the (x, y)-plane can be scaled by constant β, e.g., so that $2 \cdot \ell + d = 1$. The ratio γ is clearly invariant under such scalings. Similarly,

$$n \propto \alpha^{-1}\beta^{-2} \quad \text{and} \quad V \propto \alpha\beta^2,$$

and thus also the reduced number density $\eta = n \cdot V$ and the volume fraction ϕ are invariant.

[1]Consider a volume $V = AT$, with area A in (x, y)-plane and height T. The mean number of nodes in V is λV, i.e., the node density n is λ.

III. Analysis

In this section, we study the percolation model characterizing the dissemination of a message in space-time, estimate the the critical percolation threshold by Monte Carlo simulations, and derive an asymptotic scaling law for the threshold.

A. Directed continuum percolation

We consider a dynamic system in space-time, where each node corresponds to a (tilted) cylinder as depicted in Fig. 1. Two nodes can communicate whenever the distance between them is less than the transmission range d. The process describing the evolution of a message in the infinite plane corresponds to a *directed continuum percolation* of aligned (sheared) cylinders in three dimensions [12]. The radius r of the cylinders with respect to percolation is $d/2$, and the holding time t corresponds to the height of the cylinders. The moment two r-cylinders touch each other, messages can be transmitted. The causality imposes that the information flows only in the direction of the positive time axis.

To illustrate the directed percolation, consider a sample cluster depicted in Fig. 2 with mobile nodes. The time axis points towards the top-right corner and due to the mobility, cylinders are not aligned along the time axis. The dark red node (cylinder) indicates the source, light yellow color means that a node has not yet obtained the message, and the green color means that the node has it (at the given time). Nodes never obtaining the message have been omitted for clarity. The information "flows" only in the positive direction of time.

Formally, let $a \rightsquigarrow b$ denote that a path from node a to node b exists in space-time in the sense that a could send a message to b (potentially via a multi-hop connection). Due to the causality, $\rightsquigarrow$ is not a symmetric relation,

$$a \rightsquigarrow b \quad \not\Rightarrow \quad b \rightsquigarrow a.$$

Moreover, $\rightsquigarrow$ is not even transitive,

$$a \rightsquigarrow b \text{ and } b \rightsquigarrow c \quad \not\Rightarrow \quad a \rightsquigarrow c,$$

as a message from a to b may arrive too late for b to deliver it to c. Let $\mathcal{S}(a)$ denote the set of nodes reachable from a,

$$\mathcal{S}(a) = \{b \,:\, a \rightsquigarrow b\}.$$

We are interested in the size (i.e., the cardinality) of $\mathcal{S}(a)$, and let random variable S, *cluster size*, denote this quantity,

$$S \triangleq |\mathcal{S}(a)|.$$

When the number density of the cylinders (in space-time) is above the critical threshold, denoted by n_c, there is a positive probability that a random cylinder reaches an infinite cluster. Mathematically the critical percolation density is defined as,

$$n_c \triangleq \inf\{n \mid \mathrm{P}\{S = \infty\} > 0\}.$$

This means that with a positive propability a message get distributed "everywhere" instead of becoming extinct. The critical density is often expressed using either the critical *reduced number density* η_c or the critical *volume fraction* ϕ_c, which are both invariant to scaling as explained earlier. It depends on the shape and the alignment of the objects. For the tilted cylinders, the critical density depends on the tilt ratio, $\eta_c = \eta_c(\gamma)$. Formally,

$$\eta_c(\gamma) \triangleq \inf\{\eta \mid \mathrm{P}\{S_\gamma = \infty\} > 0\}. \tag{1}$$

Two larger sample clusters are shown in Fig. 3. The cluster on the left is finite and the message goes to extinction after some time, while the cluster on the right appears to lead to an infinite cluster, i.e., the system percolates implying[2] $\eta > \eta_c$.

The volume of the cylinders is $V = \pi r^2 t$ and the number density in space-time is $n = \lambda$. Thus, $\eta = \pi r^2 \cdot \lambda t$, yielding

$$\nu = 4\,\eta. \tag{2}$$

For the *criticality condition* we have

$$\nu_c(\gamma) = 4\,\eta_c(\gamma). \tag{3}$$

An infinite cluster with respect to the directed percolation is an infinite cluster also with respect to the undirected percolation. In [12], we already determined the critical undirected percolation threshold for aligned cylinders, $\eta_c^* = 0.3312(1)$, which thus serves as a strict lower bound for the directed percolation with $\ell = \gamma = 0$, i.e., $\eta_c(0) > \eta_c^*(0) = 0.3312(1)$.

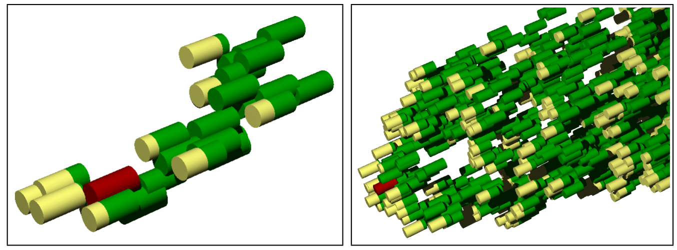
DTN network percolating in space-time means that there is a *positive probability that a message can reach destinations arbitrarily far* (in space). In this sense, the criticality condition (3) is a fundamental result for the transport capacity of DTN.

B. Methodology

An elegant way to determine the percolation threshold is based on the asymptotic behavior of the cluster size S. In particular, the tail behaves according to

$$\mathrm{P}\{S \geq s \mid \eta\} \sim A s^{2-\tau}\, f((\eta - \eta_c) s^\sigma),$$

[2]What happens when $\eta = \eta_c$ is not known in general, but for $\eta < \eta_c$, by definition, no infinite cluster emerges.



$\eta = 0.37$. $\qquad$ $\eta = 0.40$.

Figure 3: *Sample realizations of message dissemination with stationary nodes. (Left) a finite cluster of* 33 *nodes obtained with* $\eta = 0.37$. *(Right) a higher number density* $\eta = 0.40$ *enables percolation.*

where τ and σ are the so-called *universal exponents* and A is some (non-universal) constant. In three dimensions [13],

$$\begin{aligned} \tau &= 2.18906 \pm 0.00006, \\ \sigma &= 0.4522 \pm 0.0008. \end{aligned} \tag{4}$$

Near the percolation threshold, where $\eta \approx \eta_c$. The Taylor series for $f(x)$ is $f(x) = 1 + Bx + \ldots$, which gives

$$\mathrm{P}\{S \geq s \mid \eta\} \cdot s^{\tau-2} \sim A + AB(\eta - \eta_c)s^\sigma + \ldots,$$

i.e., the quantity on the left-hand side becomes a constant when $\eta = \eta_c$. The critical η_c is then determined as follows [12], [14]–[16]. First one collects a large sample set of cluster sizes and stores the results to bins so that the kth bin, denoted by B_k, corresponds to the number of samples with cluster size $s \geq 2^k$. For $\eta = \eta_c$, the quantity $B_k \cdot (2^k)^{\tau-2}$ then has a constant tail. Thus, by trial and error such η is determined.

In general, there are fewer results for the directed percolation than for the undirected case, e.g., the universal exponents (4) may be different. In Section IV, we study the directed percolation by Monte Carlo experiments and observe that the universal exponents appear to be the same.

C. Asymptotic behavior

Before proceeding with the numerical results, let us first discuss the asymptotic behavior when the distance ℓ is much longer than the transmission range d. In this case, the situation is similar to the one analyzed in [2], [18]: contacts with other nodes are point contacts, which occur at a constant rate of

$$\tilde{\lambda} \triangleq \frac{8}{\pi}\,\tilde{n} d,$$

per unit distance on the (x, y)-plane. Near the criticality threshold, only a small fraction of nodes carry the message. Assuming a node meets at most one node with the message, the contact time is uniformly distributed on interval $(0, \ell)$ along the path. In order for the system to avoid extinction, each node acquiring the message should pass it further to at least one other node on average. That is,

$$\frac{1}{\ell} \int_0^\ell s\, \frac{8}{\pi}\, \tilde{n}\, d\, ds > 1,$$

giving the fluid bound (fb),

$$\nu > \nu_c^{(\mathrm{fb})}(\gamma) \triangleq \frac{\pi^2}{4\,\gamma}, \tag{5}$$

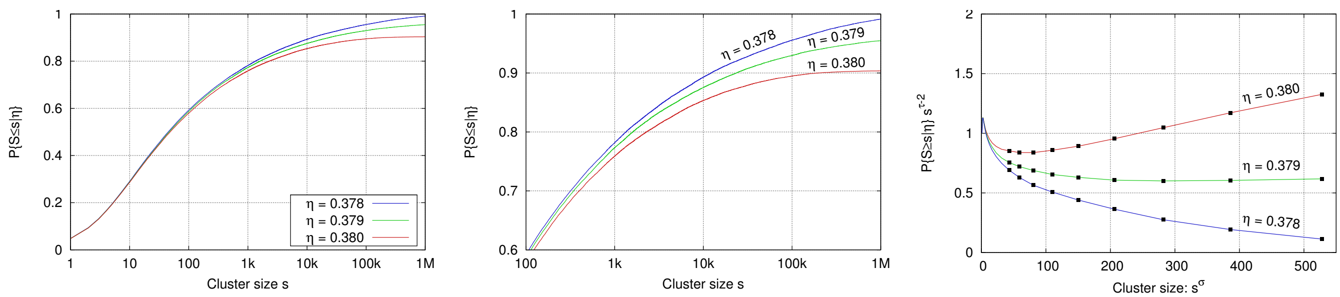


Figure 4: *Numerical results with stationary nodes,* $\gamma = 0$.

which is valid when γ tends to infinity. At this bound, $\tilde{\lambda} \cdot \ell = 2$, i.e., a node meets on average only two nodes before departing. Conversely, with aid of (2), we find a *scaling law* for the tail,

$$\eta_c(\gamma) \propto 1/\gamma, \quad \text{for } \gamma \gg 1. \tag{6}$$

IV. Numerical results

We carried out a large number of Monte Carlo experiments with a specifically developed simulation tool [17] in order to determine numerical values for $\eta_c(\gamma)$. The sample size for each (η, γ)-pair is 60000 clusters. The parameter $s_{\max}$, defining the maximum cluster size, was chosen to be $2^{20} + 1 \approx$ 1M.

A. Stationary nodes

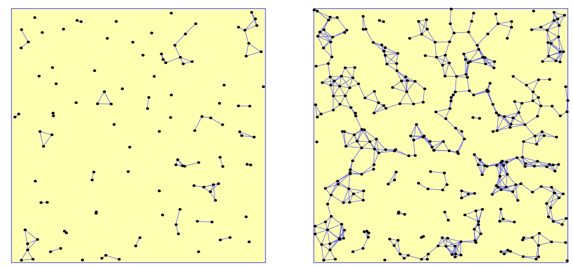
First we assume no mobility, $\gamma = 0$, and vary the reduced number density, $\eta = 0.378, \ldots, 0.380$, which corresponds to the mean node degree of $\nu = 1.512, \ldots, 1.52$. If $\nu \approx 4.51$ or higher, Gilbert's disc model percolates and a gigantic component emerges (at any given time instant) enabling multi-hop communication [1]. However, in our case the node density is clearly below that and one has to resort to DTN-style communication. This is illustrated in Fig. 5, where $\nu = 1.52$ in the left figure, and $\nu = 4.51$ in the right figure.

Recall that the percolation threshold η_c corresponds to the smallest node density at which a gigantic component emerges,

$$\eta_c = \inf\{\eta \mid \mathrm{P}\{S = \infty\} > 0\}.$$

Therefore, if CDF of the cluster size S remains asymptotically below 1, $\lim_{s\to\infty} \mathrm{P}\{S < s\} = p$ and $p < 1$, then the given system is above the percolation threshold. The quantity $1 - p$, referred to as the *strength*, is the probability that a cluster starting from a random node is infinite [9].

The simulation results are illustrated in Fig. 4. The left figure depicts CDF with a logarithmic scale on the x-axis, and the middle figure "zooms" into the interesting region. The CDF for $\eta = 0.378$ converges to 1 right after $s = 10^6$, i.e., in practice every realization becomes eventually extinct and no message will be delivered indefinitely far. However, the CDF for $\eta = 0.379$ seems to converge to a finite value less than 1, suggesting that a small fraction of messages survives forever. Increasing η further to 0.380 improves the chances of hitting a gigantic cluster to about 10%, i.e., the percolation strength is about 0.1. Consequently, our estimate for the critical density without mobility is $\eta_c(0) \approx 0.379$.



$\nu = 1.52$ $\qquad$ $\nu = 4.51$

Figure 5: *Snapshot of two networks.*

Comparing the directed percolation to the undirected one, shows that the direction constraint causes η_c to increase from $0.3312(1)$ to about 0.379, i.e., about 15% increase, in agreement with the observations made in [12].

1) Universal exponents: As mentioned, it is not clear whether the universal exponents (4) are the same in the directed case. Right figure uses the (undirected) universal exponents (4), i.e., the y-axis corresponds to $\mathrm{P}\{S > s\} \cdot s^{\tau-2}$. The curve with $\eta = 0.379$ has a constant tail, which suggests that the universal exponents indeed are the same for the undirected and directed models. More evidence to this end is given in [17].

B. Mobile nodes

Next we set $\gamma = 1$, i.e., each node moves a distance equal to the transmission range before departing. The numerical results are given in Fig. 6, which suggest that $\eta_c(1) \approx 0.2825$, as the curve with $\eta = 0.282$ appears to converge to 1 and the curve with $\eta = 0.283$ stabilizes just above 0.9. In the right graph, we again use the universal exponents and the curve with $\eta = 0.2825$ indeed has an almost constant tail.

Recall that $\nu_c = 4\,\eta_c$ according to (3). Based on our experiments, $\eta_c(1) \approx 0.75 \cdot \eta_c(0)$, which means that the node density n and the mean node degree ν can be 25% smaller already when the nodes move as little as one transmission range, i.e., the mobility improves the message forwarding capacity considerably (in space-time).

The critical reduced number density $\eta_c(\gamma)$ for several other values of γ is determined in [17] and summarized in Fig. 7. The graph depicts the behavior of the critical mean node degree $\nu_c(\gamma)$ as a function of γ. The *fluid bound* is according to

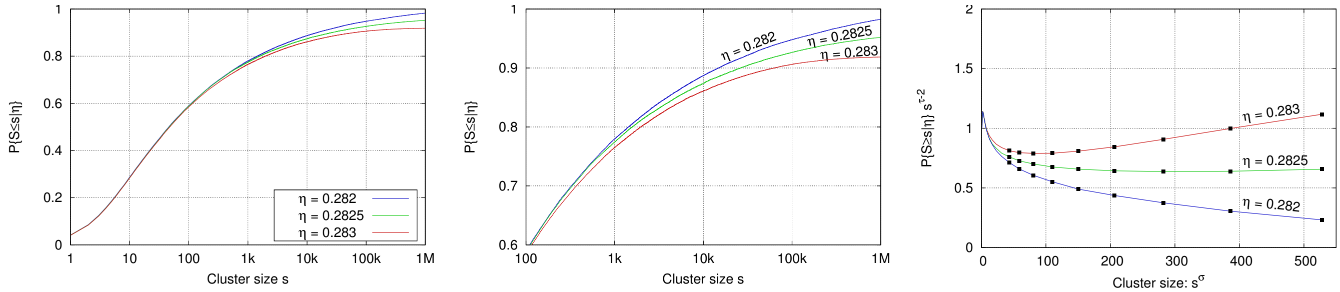


Figure 6: *Numerical results with mobile nodes, $\gamma = 1$.*

γ	$\eta_c(\gamma)$
0.0	0.3788
0.05	0.3735
0.1	0.368
0.2	0.357
0.5	0.3262
1	0.2825
2	0.2189
4	0.1459
8	0.0836
16	0.04325

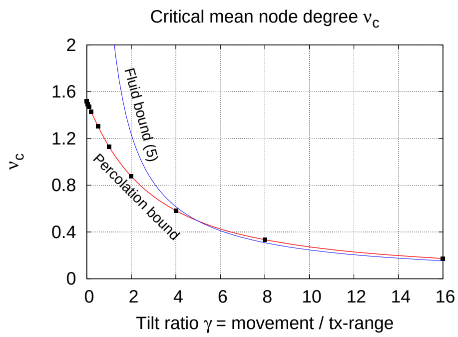


Figure 7: *Critical mean node degree $\nu_c(\gamma) = 4\,\eta_c(\gamma)$ as a function of the tilt ratio γ characterizing the relative mobility in the model. The fluid bound is according to* (5) *and valid at the limit $\gamma \to \infty$.*

(5) and valid at the limit $\gamma \to \infty$. The other curve corresponds to the critical percolation threshold, $\nu_c(\gamma) = 4\,\eta_c(\gamma)$, the values of which we have obtained numerically. The two curves differ initially, where (5) is not valid. However, for $\gamma \geq 4$ the two curves are almost matching, i.e., only for small movements one needs to analyze the situation according to the percolation, while otherwise also the fluid flow result is accurate.

V. Conclusions

DTN is designed to operate in settings where connectivity is intermittent at best. In this paper, we have analyzed necessary conditions for DTN-style communication. We assumed an elementary mobility model where nodes join the network for a constant time t and then depart. During this time, they move a distance ℓ to a random direction. We assume a large network, where the ability to sustain a message indefinitely means that the message will eventually also reach its final destination(s). This stochastic model of DTN is amenable to analysis by means of the percolation theory. In particular, we studied the so-called directed continuum percolation in space-time, where the objects are cylinders with height t and the diameter equal to the transmission range d sheared (tilted) around the time axis according to the movement ℓ. We showed that the critical percolation threshold depends only on the tilt ratio $\gamma = \ell/d$, $\eta_c = \eta_c(\gamma)$. Numerical values for $\eta_c(\gamma)$ were obtained by extensive Monte Carlo simulations. Some evidence to support the hypothesis that the universal exponents are the same for the directed and undirected models was also given. Moreover, the asymptotic behavior of $\eta_c(\gamma)$ when γ tends to infinity was derived. In terms of the mean node degree ν, our main result states that a large DTN network is operational only when $\nu > 4\,\eta_c(\gamma)$. A general observation is that increased mobility allows a lower node density, which $\eta_c(\gamma)$ then quantifies.

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