

# Random Waypoint Model in $n$ -Dimensional Space

Esa Hyytiä\* and Jorma Virtamo

*Networking Laboratory, Helsinki University of Technology,  
P.O. Box 3000, FIN 02015 HUT, Finland  
tel. +358 9 451 4786, fax. +358 9 451 2474*

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## Abstract

The random waypoint model (RWP) is one of the most widely used mobility models in performance analysis of mobile wireless networks. In this paper we extend the previous work by deriving an analytical formula for the stationary distribution of a node moving according to a random waypoint model in a  $n$ -dimensional convex set. As examples we present results for the RWP model in the unit sphere and unit box.

*Key words:* mobility modeling, random waypoint model, ad hoc networking

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## 1 Introduction

Random waypoint model (RWP) is one of the most widely used mobility models in performance analysis of wireless ad hoc networks. In the traditional RWP model in  $\mathbb{R}^2$ , the path of the node is defined by a sequence of random waypoints,  $P_1, P_2, \dots$ , placed randomly using a uniform distribution in some convex domain  $\mathcal{D} \subset \mathbb{R}^2$ . At time  $t = 0$  the node is placed at some point  $P_0 \in \mathcal{D}$ , either randomly using, e.g., uniform distribution or at some fixed starting point. Then the node moves at constant speed  $v$  along a line towards the next

waypoint  $P_1$ . Once the node reaches waypoint  $P_1$  it takes a new heading towards the next waypoint  $P_2$  etc. Each line segment between two waypoints is referred to as a leg and its length is denoted by  $\ell$ . For further details and possible extensions of the model we refer to [1].

The RWP model was originally proposed in [2] and has since then been used, e.g. in capacity and connectivity studies [3–6]. The knowledge of the stationary node distribution is often needed for determining a performance quantity of interest. The stationary node distribution in RWP model (in plane) has been studied, e.g., in [7–10] and an explicit formula for  $\mathbb{R}^2$  was derived in [1]. In this paper we extend our earlier work in [1] by deriving an analytical formula for stationary distribution of a node moving according to a RWP model in an  $n$ -dimensional convex set. The

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\* Corresponding author.

*Email addresses:* esa.hyytia@iki.fi (Esa Hyytiä), jorma.virtamo@hut.fi (Jorma Virtamo).

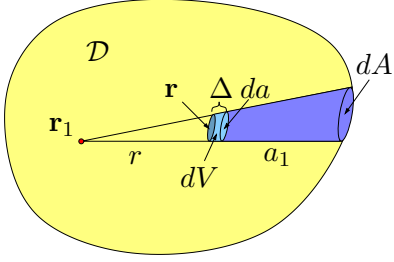


Fig. 1. The integral in Eq. (3) is equal to  $\Delta$  times the volume with darker shade.

three-dimensional RWP process serves as an elementary model for, e.g. mobile users in an office building or a shopping center.

## 2 Spatial Node Distribution in $\mathbb{R}^3$

For clarity, we first consider the natural extension of RWP model to the three-dimensional space and then generalize the results to an  $n$ -dimensional space.

Let  $V$  denote the volume of the convex domain  $\mathcal{D} \subset \mathbb{R}^3$ ,  $\ell$  the length of an arbitrary leg and  $\bar{\ell}$  the mean length of a leg. Furthermore, let  $f(\mathbf{r})$  denote the pdf of the node location at  $\mathbf{r}$ . Similarly as in [1], we start by considering a differential volume element  $dV$  at  $\mathbf{r}$  (see Fig. 1), and infer,

$$f(\mathbf{r}) = \frac{1}{\bar{\ell}} \cdot \frac{\mathbb{E}[\ell \cap dV]}{dV}, \quad (1)$$

$$\mathbb{E}[\ell \cap dV] = \frac{1}{V} \int_{\mathcal{D}} \mathbb{E}[\ell \cap dV | \mathbf{r}_1] d^3 \mathbf{r}_1. \quad (2)$$

Let  $a_1 = a_1(\mathbf{r}, \Omega)$  denote the distance from  $\mathbf{r}$  to the boundary of the domain in a given direction  $\Omega$  and  $a_2$  the distance from  $\mathbf{r}$  to the boundary in the opposite direction. Relying on Fig. 1 we have  $dV = \Delta \cdot da$  and  $da/dA = r^2/(r + a_1)^2$ .

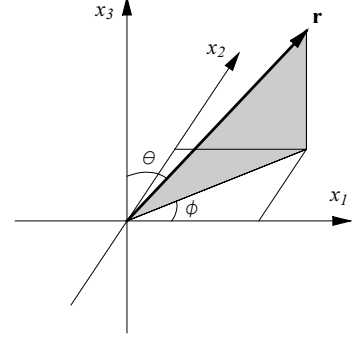


Fig. 2. Spherical Coordinates.

Consequently,

$$\begin{aligned} \mathbb{E}[\ell \cap dV | \mathbf{r}_1] &= \frac{1}{V} \int_{\mathcal{D}} |\ell(\mathbf{r}_1, \mathbf{r}_2) \cap dV| d^3 \mathbf{r}_2 \\ &= \frac{1}{3V} \Delta \cdot ((r + a_1) \cdot dA - r \cdot da) \\ &= \frac{\Delta}{3V} \cdot \left( \frac{(r + a_1)^3 - r^3}{r^2} \right) da \\ &= \frac{dV}{3V} \cdot \frac{(r + a_1)^3 - r^3}{r^2}. \end{aligned} \quad (3)$$

Substituting Eqs. (2) and (3) back into Eq. (1) yields,

$$\begin{aligned} f(\mathbf{r}) &= \frac{1}{3\bar{\ell}V^2} \int_{\mathcal{D}} \frac{(r + a_1)^3 - r^3}{r^2} d^3 \mathbf{r}_1 \\ &= \frac{1}{3\bar{\ell}V^2} \int_{\Omega} d\Omega \int_0^{a_2} dr \left( (r + a_1)^3 - r^3 \right) \\ &= \frac{1}{12\bar{\ell}V^2} \int_{\Omega} d\Omega \left( (a_1 + a_2)^4 - (a_1^4 + a_2^4) \right). \end{aligned}$$

More specifically, using spherical coordinates we get (see Fig. 2)

$$f(\mathbf{r}) = \frac{1}{6\bar{\ell}V^2} \int_0^\pi d\theta \sin \theta \int_0^\pi d\phi H(\mathbf{r}, \theta, \phi). \quad (4)$$

where  $H(\mathbf{r}, \theta, \phi) = (a_1 + a_2)^4 - (a_1^4 + a_2^4)$ .

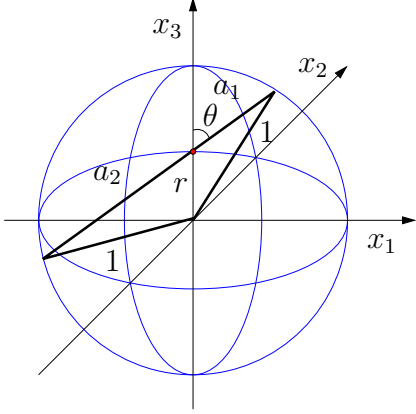


Fig. 3. Due to symmetry in a unit sphere one can consider points  $(0, 0, r)$  for which  $a_1$  and  $a_2$  are a function of  $r$  and  $\theta$ .

### 3 Spatial Node Distribution in $\mathbb{R}^n$

The same procedure can be generalized to  $n$  dimensions. The “volume” of an  $n$ -dimensional cone is

$$V_n(h) = \frac{h \cdot A}{n},$$

where  $h$  is the height and  $A$  corresponds to the “area” of the base. Let  $\mathcal{D} \subset \mathbb{R}^n$  be a convex set with volume  $V$ . Then it is easy to see that the stationary distribution of the RWP process in  $\mathcal{D}$  is given by

$$f(\mathbf{r}) = \frac{1}{n\bar{\ell}V^2} \int_{\mathcal{D}} \frac{(r + a_1)^n - r^n}{r^{n-1}} d^n \mathbf{r},$$

which can be expressed as

$$f(\mathbf{r}) = \frac{1}{n(n+1)V^2\bar{\ell}} \int_{\Omega} d\Omega H(\mathbf{r}, \Omega). \quad (5)$$

where

$$H(\mathbf{r}, \Omega) = (a_1 + a_2)^{n+1} - (a_1^{n+1} + a_2^{n+1}).$$

The mean length of leg  $\bar{\ell}$  is obtained from the normalization condition  $\int f(\mathbf{r}) d^n \mathbf{r} = 1$ ,

$$\bar{\ell} = \frac{1}{n(n+1)V^2} \int_{\mathcal{D}} d^n \mathbf{r} \int_{\Omega} d\Omega H(\mathbf{r}, \Omega). \quad (6)$$

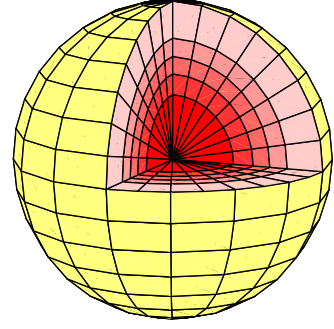


Fig. 4. Stationary node distribution of the RWP process in the unit sphere. Each section represents a probability mass of 0.2.

### 4 Examples

*Example 1: RWP model in plane:* For  $n = 2$  the general expression (5) yields

$$\begin{aligned} f(\mathbf{r}) &= \frac{1}{6\bar{\ell}V^2} \int_0^{2\pi} (a_1 + a_2)^3 - (a_1^3 + a_2^3) d\theta \\ &= \frac{1}{\bar{\ell}V^2} \int_0^\pi a_1 a_2 (a_1 + a_2) d\theta, \end{aligned}$$

which is identical to the equation derived in [1].

*Example 2: Unit sphere in  $\mathbb{R}^3$ :* Due to the symmetry, the pdf is a function of distance  $r = |\mathbf{r}|$  only and without loss of generality we can consider the point  $\mathbf{r} = (0, 0, r)$ . From Fig. 3 one immediately obtains that

$$\begin{aligned} a_1(r, \theta) &= \sqrt{1 - r^2 \sin^2 \theta} - r \cdot \cos \theta, \\ a_2(r, \theta) &= \sqrt{1 - r^2 \sin^2 \theta} + r \cdot \cos \theta. \end{aligned} \quad (7)$$

Using the 3-dimensional expression (4) we get

$$f(\mathbf{r}) = \frac{3}{16\pi \cdot \bar{\ell}} \int_0^\pi \overbrace{\sin \theta \cdot a_1 a_2 (2a_1^2 + 3a_1 a_2 + 2a_2^2)}^{h(r, \theta)} d\theta.$$

Writing

$$h(r, \theta) = (1 - r^2)(7 - 3r^2 + 4r^2 \cos 2\theta) \sin \theta$$

and

$$h(r) = \int_0^\pi h(r, \theta) d\theta = \frac{2}{3} (21 - 34r^2 + 13r^4)$$

we get

$$\int_0^1 4\pi r^2 \cdot h(r) dr = \frac{192\pi}{35}.$$

Thus the mean length of leg  $\bar{\ell}$  and the pdf of the node location at  $\mathbf{r}$  are

$$\bar{\ell} = \frac{36}{35} \approx 1.029,$$

$$f(\mathbf{r}) = \frac{35}{288\pi} \cdot (21 - 34r^2 + 13r^4).$$

In the spherically symmetric case, the pdf of the random variable  $r$ , denoted by  $f_d(r)$ , is  $f_d(r) = 4\pi r^2 f(\mathbf{r})$ ,

$$f_d(r) = \frac{35}{72} \cdot r^2 (21 - 34r^2 + 13r^4).$$

The cumulative pdf for the unit sphere is illustrated in Fig. 4, where each section represents a probability mass of 0.2.

*Example 3: Unit box in  $\mathbb{R}^3$ :* The next example is the unit box in 3-dimensional space. The pdf of the node location can be evaluated numerically using Eq. (4). In Fig. 5 the pdf is depicted for different values of the  $x_1$ -coordinate. Fig. 6 illustrates the marginal distribution of the node location (solid line) and the pdf of the node location in one-dimensional RWP (dashed line). It can be noted that the difference in pdfs is almost negligible.

*Example 4: Unit hypersphere in  $\mathbb{R}^n$ :* The area and volume of the  $n$ -dimensional unit hyper-

sphere are [11]

$$A_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \quad \text{and} \quad V_n = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}.$$

Again, due to the symmetry, the pdf is a function of distance  $r = |\mathbf{r}|$  only and we can arbitrarily take  $\mathbf{r} = (0, \dots, 0, r)$ . Eq. (7) holds still with  $\theta$  being the angle between  $\mathbf{r}$  and the  $x_n$ -axis. Differential surface area of the hypersphere between  $\theta$  and  $\theta + d\theta$  is  $A_{n-1} \sin^{n-2} \theta d\theta$ . Hence, we have

$$f(\mathbf{r}) = \frac{A_{n-1}}{n(n+1)V_n^2 \cdot \bar{\ell}} \cdot h(r),$$

where

$$h(r) = \int_0^\pi \sin^{n-2} \theta \cdot H(r, \theta) d\theta,$$

$$H(r, \theta) = ((a_1 + a_2)^{n+1} - (a_1^{n+1} + a_2^{n+1})).$$

In Fig. 7 the pdf of the node location at  $\mathbf{r}$ ,  $f(\mathbf{r})$ , and the pdf of the distance  $r$  from the origin,  $f_d(r)$ , are depicted for dimensions  $n = 2, 3, 4, 5$ . The maximum value of the pdf  $f(r)$  attained at the center of the hypersphere increases as the dimension  $n$  increases. Furthermore, from the Fig. 7(b) it can be noted that as the dimensions increase the probability mass

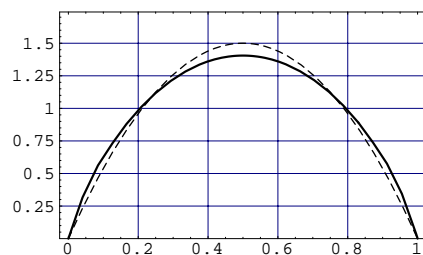


Fig. 6. Marginal pdf of the node location in the unit box (solid line) compared with the pdf of the node location in 1-dimensional RWP process (dashed line).

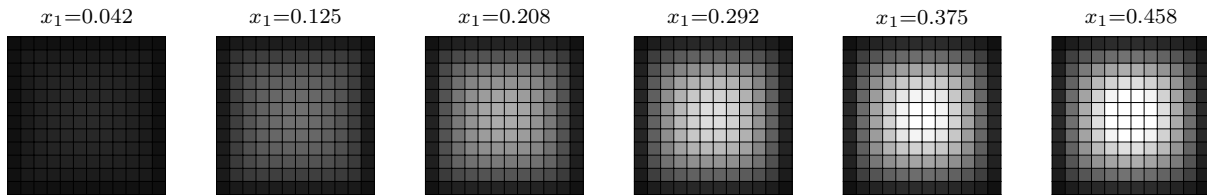


Fig. 5. Pdf of the node location in the unit box at different “slices” of  $x_1$ .

of the random variable  $r$  shifts towards the surface,  $r = 1$ . Finally, the mean length of leg given by Eq. (6) can be written for the unit hypersphere as

$$\bar{\ell}_n = \frac{A_{n-1}A_n}{n(n+1)V_n^2} \int_0^1 r^{n-1}h(r) dr.$$

For  $n = 1, \dots, 5$  we have explicitly

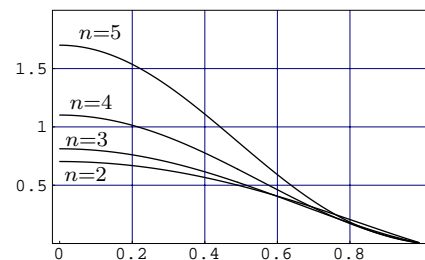
$$\begin{aligned} \bar{\ell}_1 &= 2/3 && \approx 0.667, \\ \bar{\ell}_2 &= 128/45\pi && \approx 0.905, \\ \bar{\ell}_3 &= 36/35 && \approx 1.029, \\ \bar{\ell}_4 &= 16384/4725\pi && \approx 1.104, \\ \bar{\ell}_5 &= 800/693 && \approx 1.154. \end{aligned}$$

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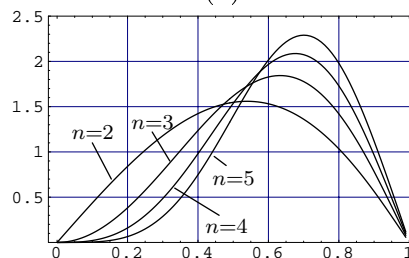
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(a)



(b)

Fig. 7. Pdf of the node location (left) and distance from the origin (right) in an  $n$ -dimensional unit hypersphere,  $n = 2, 3, 4, 5$ .

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**Esa Hyytiä** received the M.Sc. (Tech.) degree in engineering physics and Lic.Sc. (Tech.) degree in electrical engineering from the Helsinki University of Technology, in 1998 and 2001, respectively. He is currently pursuing the D.Sc. (Tech.) degree at the Networking Laboratory of Helsinki University of Technology. His research interests include performance analysis, design and optimization of optical and ad hoc networks.



**Jorma Virtamo** received the M.Sc. (Tech.) degree in engineering physics and D.Sc. (Tech.) degree in theoretical physics from the Helsinki University of Technology, in 1970 and 1976, respectively. In 1986, he joined the Technical Research Centre of Finland, VTT Information Technology, where he led a teletraffic research group, and became a Research Professor in 1995. Since 1997 he has been a Professor in the Networking Laboratory of the Helsinki University of Technology. His current research interests include queueing theory and performance analysis and control of the Internet, optical networks and ad hoc networks.