Birth-death processes

General

A birth-death (BD process) process refers to a Markov process with

- a discrete state space
- the states of which can be enumerated with index $i=0,1,2,...$ such that
- state transitions can occur only between neighbouring states, $i \rightarrow i + 1$ or $i \rightarrow i - 1$

Transition rates

$$q_{i,j} = \begin{cases} 
\lambda_i & \text{when } j = i + 1 \\
\mu_i & \text{if } j = i - 1 \\
0 & \text{otherwise}
\end{cases}$$

Probability of birth in interval $\Delta t$ is $\lambda_i \Delta t$

Probability of death in interval $\Delta t$ is $\mu_i \Delta t$

when the system is in state $i$
The equilibrium probabilities of a BD process

We use the method of a cut = global balance condition applied on the set of states 0, 1, . . . , k.

In equilibrium the probability flows across the cut are balanced (net flow =0)

$$\lambda_k \pi_k = \mu_{k+1} \pi_{k+1} \quad k = 0, 1, 2, \ldots$$

We obtain the recursion

$$\pi_{k+1} = \frac{\lambda_k}{\mu_{k+1}} \pi_k$$

By means of the recursion, all the state probabilities can be expressed in terms of that of the state 0, \(\pi_0\),

$$\pi_k = \frac{\lambda_{k-1} \lambda_{k-2} \cdots \lambda_0}{\mu_k \mu_{k-1} \cdots \mu_1} \pi_0 = \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}} \pi_0$$

The probability \(\pi_0\) is determined by the normalization condition \(\pi_0\)

$$\pi_0 = \frac{1}{1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \cdots} = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}}$$
The time-dependent solution of a BD process

Above we considered the equilibrium distribution $\pi$ of a BD process.

Sometimes the state probabilities at time 0, $\pi(0)$, are known
- usually one knows that the system at time 0 is precisely in a given state $k$; then $\pi_k(0) = 1$ and $\pi_j(0) = 0$ when $j \neq k$

and one wishes to determine how the state probabilities evolve as a function of time $\pi(t)$
- in the limit we have $\lim_{t \to \infty} \pi(t) = \pi$.

This is determined by the equation

$$\frac{d}{dt}\pi(t) = \pi(t) \cdot Q$$

where

$$Q = \begin{pmatrix}
-\lambda_0 & \lambda_0 & 0 & \ldots & \ldots \\
\mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \ldots \\
0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 \\
\vdots & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 \\
\vdots & \vdots & 0 & \mu_4 & -(\lambda_4 + \mu_4)
\end{pmatrix}$$
The time-dependent solution of a BD process (continued)

The equations component wise

\[
\begin{align*}
\frac{d\pi_i(t)}{dt} &= \underbrace{- (\lambda_i + \mu_i)\pi_i(t)}_{\text{flows out}} + \underbrace{\lambda_{i-1}\pi_{i-1}(t)}_{\text{flows in}} + \underbrace{\mu_{i+1}\pi_{i+1}(t)}_{\text{flows in}} \\
\frac{d\pi_0(t)}{dt} &= \underbrace{-\lambda_0\pi_0(t)}_{\text{flow out}} + \underbrace{\mu_1\pi_1(t)}_{\text{flow in}}
\end{align*}
\]
Example 1. Pure death process

\[
\begin{cases}
    \lambda_i = 0 & i = 0, 1, 2, \\ 
    \mu_i = i\mu & \\
\end{cases}
\]

\[\pi_i(0) = \begin{cases}
    1 & i = n \\
    0 & i \neq n
\end{cases}\]

all individuals have the same mortality rate \(\mu\)

the system starts from state \(n\)

State 0 is an absorbing state, other states are transient

Binomial distribution: the survival probability at time \(t\) is \(e^{-\mu t}\) independent of others
Example 2. Pure birth process (Poisson process)

\[
\begin{cases}
\lambda_i = \lambda \\
\mu_i = 0
\end{cases} \quad i = 0, 1, 2, \ldots
\]

\[
\pi_i(0) = \begin{cases}
1 & i = 0 \\
0 & i > 0
\end{cases}
\]

Birth probability per time unit is constant \(\lambda\).

Initially the population size is 0.

\[\begin{array}{cccccc}
0 & \lambda & 1 & \lambda & \cdots & \lambda & i-1 & \lambda & i & \lambda \\
\end{array}\]

All states are transient.

\[
\begin{aligned}
\frac{d}{dt} \pi_i(t) &= -\lambda \pi_i(t) + \lambda \pi_{i-1}(t) & i > 0 \\
\frac{d}{dt} \pi_0(t) &= -\lambda \pi_0(t) & \Rightarrow & \pi_0(t) = e^{-\lambda t} \\
\frac{d}{dt} (e^{\lambda t} \pi_i(t)) &= \lambda \pi_{i-1}(t)e^{\lambda t} & \Rightarrow & \pi_i(t) = e^{-\lambda t} \lambda \int_0^t \pi_{i-1}(t')e^{\lambda t'} dt' \\
\pi_1(t) &= e^{-\lambda t} \lambda \int_0^t \frac{e^{-\lambda t'}e^{\lambda t'}}{1} dt' = e^{-\lambda t}(\lambda t) \\
\end{aligned}
\]

Recursively

\[
\pi_i(t) = \frac{(\lambda t)^i}{i!}e^{-\lambda t} 
\]

Number of births in interval \((0, t) \sim \text{Poisson}(\lambda t)\)
Example 3. A single server system

- constant arrival rate $\lambda$ (Poisson arrivals)
- stopping rate of the service $\mu$ (exponential distribution)

The states of the system

$$
\begin{align*}
\pi_0(t) &= -\lambda \pi_0(t) + \mu \pi_1(t) \\
\pi_1(t) &= \lambda \pi_0(t) - \mu \pi_1(t)
\end{align*}
$$

$$
Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}
$$

BY adding both sides of the equations

$$
\frac{d}{dt} \pi_0(t) + \pi_1(t)) = 0 \quad \Rightarrow \quad \pi_0(t) + \pi_1(t) = \text{constant} = 1 \quad \Rightarrow \quad \pi_1(t) = 1 - \pi_0(t)
$$

$$
\frac{d}{dt} \pi_0(t) + (\lambda + \mu) \pi_0(t) = \mu \quad \Rightarrow \quad \frac{d}{dt} (e^{(\lambda+\mu)t} \pi_0(t)) = \mu e^{(\lambda+\mu)t}
$$

$$
\begin{align*}
\pi_0(t) &= \frac{\mu}{\lambda + \mu} + (\pi_0(0) - \frac{\mu}{\lambda + \mu}) e^{-\lambda t} \\
\pi_1(t) &= \frac{\lambda}{\lambda + \mu} + (\pi_1(0) - \frac{\lambda}{\lambda + \mu}) e^{-\mu t}
\end{align*}
$$
Summary of the analysis on Markov processes

1. Find the state description of the system
   - no ready recipe
   - often an appropriate description is obvious
   - sometimes requires more thinking
   - a system may Markovian with respect to one state description but non-Markovian with respect to another one (the state information is not sufficient)
   - finding the state description is the creative part of the problem

2. Determine the state transition rates
   - a straightforward task when holding times and interarrival times are exponential

3. Solve the balance equations
   - in principle straightforward (solution of a set of linear equations)
   - the number of unknowns (number of states) can be very great
   - often the special structure of the transition diagram can be exploited
Global balance

\[ \sum_{j \neq i} \pi_j q_{j,i} = \sum_{j \neq i} \pi_i q_{i,j} \]

flow to state \( i \)
flow out of state \( i \)

\[ i = 0, 1, \ldots, n \]

one equation per each state

\[ n + 1 \text{ states} \]

\[ \pi \cdot Q = 0 \]

one equation is redundant

\[ \pi_0 + \pi_1 + \cdots + \pi_n = 1 \]

normalization condition
Example 1. A queueing system

The number of customers in system $N$ is an appropriate state variable

- uniquely determines the number of customers in service and in waiting room
- after each arrival and departure the remaining service times of the customers in service are $\text{Exp}(\mu)$ distributed (memoryless)
Example 2. Call blocking in an ATM network

A virtual path (VP) of an ATM network is offered calls of two different types.

\[
\begin{align*}
R_1 &= 1\text{Mbps} \\
\lambda_1 &= \text{arrival rate} \\
\mu_1 &= \text{mean holding time} \\
R_2 &= 2\text{Mbps} \\
\lambda_2 &= \text{arrival rate} \\
\mu_2 &= \text{mean holding time}
\end{align*}
\]

a) The capacity of the link is large (infinite)

The state variable of the Markov process in this example is the pair \((N_1, N_2)\), where \(N_i\) defines the number of class-\(i\) connections in progress.
Call blocking in an ATM network (continued)

b) The capacity of the link is 4.5 Mbps