HELSINKI UNIVERSITY OF TECHNOLOGY | **Exercise 2**
Networking Laboratory | 13.11.2007
S-38.3143 Queueing Theory, II/2007 | Virtamo / Penttinen

1. Let $S = X_1 + \ldots + X_N$, where $X_i \sim \mathrm{Exp}(\mu)$, be i.i.d. and N an independent geometrically distributed random variable, $\mathrm{P}\{N = k\} = (1-p)p^{k-1}$, $k = 1, 2, \ldots$. Determine the tail distribution of S, $G(x) = \mathrm{P}\{S > x\}$.

2. Prove (without using the generating function), that the sum of two Poisson random variables, $N_1 \sim$ Poisson(a_1) and $N_2 \sim$ Poisson(a_2), is also Poisson distributed: $(N_1 + N_2) \sim$ Poisson($a_1 + a_2$). Prove the same result with the aid of generating functions.

3. Let X be an exponential random variable. Without any computations, tell which one of the following is correct. Explain your answer.

   a) $\mathrm{E}[X^2 | X > 1] = \mathrm{E}[(X+1)^2]$
   b) $\mathrm{E}[X^2 | X > 1] = \mathrm{E}[X^2] + 1$
   c) $\mathrm{E}[X^2 | X > 1] = (1 + \mathrm{E}[X])^2$

4. The transition probability matrix of a four state Markov chain is

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

   Draw the state transition diagram of the chain and deduce which states are transient and which are recurrent.

5. Time to absorption. A three state Markov chain (with states $i = 1, \ldots, 3$) has the state transition probability matrix:

$$\mathbf{P} = \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1/4 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}.$$

   State 3 is an absorbing state. Let T_i denote the average time (number of steps) needed by a system in state i to reach the absorbing state 3 ($T_3 = 0$). Write equations for T_i, $i = 1, 2$, based on the facts that the next transition $i \to j$ takes one step and due to the Markovian property it takes T_j steps on the average to reach the absorbing state from state j. Solve the equations.

6. Define the state of the system at the n^{th} trial of an infinite sequence of Bernoulli(p) trials to be the number of consecutive succesful trials preceeding and the current trial, i.e. the state is the distance to previous unsuccesful trial. If the n^{th} trial is unsuccesful, then $X_n = 0$; if it succeeds but the previous one was unsuccesful, then $X_n = 1$, etc. a) What is the state space of the system? b) Argue that X_n forms a Markov chain. c) Write down the transition probability matrix of the Markov chain (give its structure).