HELSINKI UNIVERSITY OF TECHNOLOGY
Networking Laboratory
S-38.3143 Queueing Theory, Fall 2005

Exercise 6
13.12.2005
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1. Carloads of customers arrive at a single-server station in accordance with a Poisson process with rate 4 per hour. The service times are exponentially distributed with mean 3 min. If each carload contains either 1, 2, or 3 customers with respective probabilities $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$, compute the average customer waiting time in the queue. Hint: The waiting time of the first customer of each group can be obtained from an appropriate $M/G/1$ queue. Consider separately the "internal" waiting time in the group.

2. The Pollaczek-Khinchin formula for the Laplace transform of the waiting time W is

$$W^*(s) = \frac{s(1-\rho)}{s - \lambda + \lambda S^*(s)}$$

where $S^*(s)$ is the Laplace transform of the service time S and $\rho = \lambda\overline{S}$. Using this result, rederive the PK mean formula for the waiting time.

3. Let $B^*(s)$ denote the Laplace transform of the busy period B in an $M/G/1$ queue. This satisfies so-called Takács' equation,

$$B^*(s) = S^*(s + \lambda - \lambda B^*(s)),$$

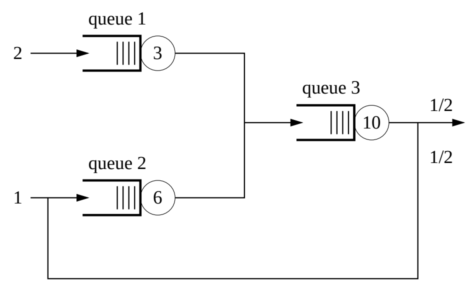
where $S^*(s)$ is the Laplace transform of the service time S.

a) Derive expression for $B^*(s)$ in the case of an $M/M/1$ queue, i.e. when $S^*(s) = \mu/(\mu + s)$.

b) Derive the following expressions for the first two moments of B:

$$\overline{B} = \frac{\overline{S}}{1-\rho}, \qquad \overline{B^2} = \frac{\overline{S^2}}{(1-\rho)^3}.$$

4. Queues 1 and 2 of the open Jackson queueing network depicted in the figure receive Poissonian arrival streams with rates 2 and 1 (customers/s). Service times are exponentially distributed with the given rates (customers/s). Calculate a) customer streams through each of the queues, b) average occupancies of the queues and the average total number of customers in the network, c) mean delays in the network of customers arriving at queues 1 and 2 as well as the delay of an arriving customer chosen at random.



5. A closed queueing network consists of three queues in a ring. The service rates of the queues are μ, 2μ and 4μ. There are two customers circulating in the ring. Find the mean queue lengths of the queues and the mean round trip time of a customer.

6. Burke's own proof for his theorem: let T_1 and T_2 be two successive service completion time instants in an M/M/1 queue and let $G_j(t)$ denote the joint probability that there are j customers in the system at time $T_1 + t$ and that $T_2 > T_1 + t$.

   a) Derive a system of differential equations for the functions $G_j(t)$ and show that with the initial condition $G_j(0) = \pi_j^*$, the solution is $G_j(t) = \pi_j^* e^{-\lambda t}$, where π_j^* is the probability that immediately after an arbitrary departure, there are j customers in the system (it has been shown in the lectures that π_j^* obeys an equilibrium distribution, $\pi_j^* = \pi_j$). Based on this, show that the interval $T_2 - T_1$ is exponentially distributed with parameter λ.

   b) Show that the length of the interval $T_2 - T_1$ and the queue length N_{T_2+} right after the instant T_2 are independent, and further that the various departure intervals are independent of each other and N_t is independent of departure instants before time t. Hint: prove that the probability of the event $\{T_2 - T_1 \in (t, t + dt)$ and $N_{T_2+} = j\}$ is $G_{j+1}(t)\mu dt$ and utilize the fact that N_t is a Markov process.