HELSINKI UNIVERSITY OF TECHNOLOGY
Networking Laboratory
S-38.3143 Queueing Theory, Fall 2005

Exercise 2
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1. The transition probability matrix of a four state Markov chain is

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

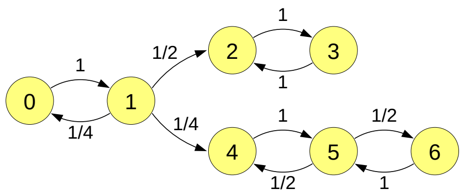
   Draw the state transition diagram of the chain and deduce which states are transient and which are recurrent.

2. Markov chain has the following state diagram:

   a) Classify the states.

   b) Determine class specific equilibrium probabilities of positively recurrent states.

   c) Calculate numerically $\lim_{n\to\infty} \mathbf{P}^n$ using a matrix multiplication. What do you observe?



3. Time to absorption. A three state Markov chain (with states $i = 1, \ldots, 3$) has the state transition probability matrix:

$$\mathbf{P} = \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1/4 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}.$$

   State 3 is an absorbing state. Let T_i denote the average time (number of steps) needed by a system in state i to reach the absorbing state 3 ($T_3 = 0$). Write equations for T_i, $i = 1, 2$, based on the facts that the next transition $i \to j$ takes one step and due to the Markovian property it takes T_j steps on the average to reach the absorbing state from state j. Solve the equations.

4. You have a box with four balls of different colors. At each step two balls are picked randomly and then the first ball is painted to match the color of the second ball. What is the expected number of steps before all the balls are of the same color?

5. A Markov chain has the following transition probability matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$$

   a) Determine the state probability vector $\boldsymbol{\pi}$ in equilibrium.

   b) Determine the generating function (matrix) $(\mathbf{I} - z\mathbf{P})^{-1}$ of matrix powers $\mathbf{P}^n$, $n = 0, 1, \ldots$.

   c) From the previous result, deduce the general form of $\mathbf{P}^n$? (Hint: write the generating function as an expression in terms of z, where the coefficients are matrices independent of z. Then apply partial fraction decomposition.) Deduce again the state probabilities distribution in equilibrium.

6. A group of n customers moves around among two queues in a closed system. Upon completion of service, the served customer then joins the other queue (or enters service if the server is free). All service times are exponential with rate μ. Find the proportion of time that there are j customers at server 1, $j = 0, \ldots, n$.