5. Stochastic processes (1)

Stochastic processes (1)

- Consider some quantity in a teletraffic (or any) system
- It typically evolves in time randomly
  - Example 1: the number of occupied channels in a telephone link at time \( t \) or at the arrival time of the \( n \)th customer
  - Example 2: the number of packets in the buffer of a statistical multiplexer at time \( t \) or at the arrival time of the \( n \)th customer
- This kind of evolution is described by a stochastic process
  - At any individual time \( t \) (or \( n \)) the system can be described by a random variable
  - Thus, the stochastic process is a collection of random variables

Stochastic processes (2)

- **Definition:** A (real-valued) stochastic process \( X = (X_t | t \in I) \) is a collection of random variables \( X_t \)
  - taking values in some (real-valued) set \( S \), \( X(\omega) \in S \), and
  - indexed by a real-valued (time) parameter \( t \in I \).
- Stochastic processes are also called random processes (or just processes)
- The index set \( I \subset \Re \) is called the parameter space of the process
- The value set \( S \subset \Re \) is called the state space of the process
- Note: Sometimes notation \( X_t \) is used to refer to the whole stochastic process (instead of a single random variable related to the time \( t \))
Stochastic processes (3)

- Each (individual) random variable $X_t$ is a mapping from the sample space $\Omega$ into the real values $\mathbb{R}$:

$$X_t : \Omega \rightarrow \mathbb{R}, \ \omega \mapsto X_t(\omega)$$

- Thus, a stochastic process $X$ can be seen as a mapping from the sample space $\Omega$ into the set of real-valued functions $\mathbb{R}^I$ (with $t \in I$ as an argument):

$$X : \Omega \rightarrow \mathbb{R}^I, \ \omega \mapsto X(\omega)$$

- Each sample point $\omega \in \Omega$ is associated with a real-valued function $X(\omega)$. Function $X(\omega)$ is called a realization (or a path or a trajectory) of the process.

Example

- Consider traffic process $X = (X_t | t \in [0,T])$ in a link between two telephone exchanges during some time interval $[0,T]$.
  - $X_t$ denotes the number of occupied channels at time $t$.
- Sample point $\omega \in \Omega$ tells us:
  - what is the number $X_0$ of occupied channels at time $0$,
  - what are the remaining holding times of the calls going on at time $0$,
  - at what times new calls arrive, and
  - what are the holding times of these new calls.
- From this information, it is possible to construct the realization $X(\omega)$ of the traffic process $X$.
  - Note that all the randomness in the process is included in the sample point $\omega$.
  - Given the sample point, the realization of the process is just a (deterministic) function of time.

Summary

- Given the sample point $\omega \in \Omega$:
  - $X(\omega) = (X_t(\omega) | t \in I)$ is a real-valued function of $t \in I$.
- Given the time index $t \in I$,
  - $X_t(\omega) = (X_t(\omega) | \omega \in \Omega)$ is a random variable (as $\omega \in \Omega$).
- Given the sample point $\omega \in \Omega$ and the time index $t \in I$,
  - $X_t(\omega)$ is a real value.
Categories of stochastic processes

- Reminder:
  - Parameter space: set $I$ of indices $t \in I$
  - State space: set $S$ of values $X_t(\omega) \in S$

- Categories:
  - Based on the parameter space:
    - Discrete-time processes: parameter space discrete
    - Continuous-time processes: parameter space continuous
  - Based on the state space:
    - Discrete-state processes: state space discrete
    - Continuous-state processes: state space continuous

- In this course we will concentrate on the discrete-state processes (with either a discrete or a continuous parameter space (time))
  - Typical processes describe the number of customers in a queueing system (the state space being thus $S = \{0, 1, 2, \ldots\}$)

Examples

- Discrete-time, discrete-state processes
  - Example 1: the number of occupied channels in a telephone link at the arrival time of the $n^{th}$ customer, $n = 1, 2, \ldots$
  - Example 2: the number of packets in the buffer of a statistical multiplexer at the arrival time of the $n^{th}$ customer, $n = 1, 2, \ldots$

- Continuous-time, discrete-state processes
  - Example 3: the number of occupied channels in a telephone link at time $t > 0$
  - Example 4: the number of packets in the buffer of a statistical multiplexer at time $t > 0$

Notation

- For a discrete-time process,
  - the parameter space is typically the set of positive integers, $I = \{1, 2, \ldots\}$
  - Index $t$ is then (often) replaced by $n$: $X_n, X_n(\omega)$

- For a continuous-time process,
  - the parameter space is typically either a finite interval, $I = [0, T]$, or all non-negative real values, $I = [0, \infty)$
  - In this case, index $t$ is (often) written not as a subscript but in parentheses: $X(t), X(t(\omega))$

Distribution

- The stochastic characterization of a stochastic process $X$ is made by giving all possible finite-dimensional distributions

\[ P\{X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n\} \]

where $t_1, \ldots, t_n \in I$, $x_1, \ldots, x_n \in S$ and $n = 1, 2, \ldots$

- In general, this is not an easy task because of dependencies between the random variables $X_t$ (with different values of time $t$)

- For discrete-state processes it is sufficient to consider probabilities of the form

\[ P\{X_{t_1} = x_1, \ldots, X_{t_n} = x_n\} \]

  - cf. discrete distributions
5. Stochastic processes (1)

### Dependence

- The most simple (but not so interesting) example of a stochastic process is such that all the random variables $X_t$ are independent of each other. In this case

$$P\{X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n\} = P\{X_{t_1} \leq x_1\} \cdots P\{X_{t_n} \leq x_n\}$$

- The most simple non-trivial example is a discrete state Markov process. In this case

$$P\{X_t = x_1\} \cdot P\{X_{t_1} = x_2 \mid X_t = x_1\} \cdots P\{X_n = x_n \mid X_{t_{n-1}} = x_{n-1}\}$$

- This is related to the so called Markov property:
  - Given the current state (of the process), the future (of the process) does not depend on the past (of the process), i.e. how the process has arrived to the current state

### Stationarity

- **Definition:** Stochastic process $X$ is **stationary** if all finite-dimensional distributions are invariant to time shifts, that is:

$$P\{X_{t_1+\Delta} \leq x_1, \ldots, X_{t_n+\Delta} \leq x_n\} = P\{X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n\}$$

for all $\Delta, n, t_1, \ldots, t_n$ and $x_1, \ldots, x_n$

- **Consequence:** By choosing $n = 1$, we see that all (individual) random variables $X_t$ of a stationary process are identically distributed:

$$P\{X_t \leq x\} = F(x)$$

for all $t \in I$. This is called the **stationary distribution** of the process.

### Arrival process

- An arrival process can be described as
  - a **point process** $(\tau_n \mid n = 1,2,\ldots)$ where $\tau_n$ tells the arrival time of the $n$th customer (discrete-time, continuous-state)
    - non-decreasing: $\tau_n \geq \tau_{n-1}$ kaikilla $n$
      - thus non-stationary!
    - typically it is assumed that the interarrival times $\tau_n - \tau_{n-1}$ are independent and identically distributed (IID) ⇒ renewal process
    - then it is sufficient to specify the interarrival time distribution
    - exponential IID interarrival times ⇒ Poisson process
  - a **counter process** $(A(t) \mid t \geq 0)$ where $A(t)$ tells the number of arrivals up to time $t$ (continuous-time, discrete-state)
    - non-decreasing: $A(t+\Delta) \geq A(t)$ for all $t, \Delta \geq 0$
      - thus non-stationary!
    - independent and identically distributed (IID) increments $A(t+\Delta) - A(t)$ with Poisson distribution ⇒ Poisson process

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5. Stochastic processes (1)

### Stochastic processes in teletraffic theory

- In this course (and, more generally, in teletraffic theory) various stochastic processes are needed to describe
  - the arrivals of customers to the system (arrival process)
  - the state of the system (state process)

- Note that the latter is also often called as traffic process
### State process

- **In simple cases**
  - the state of the system is described just by an integer
  - e.g. the number $X(t)$ of calls or packets at time $t$
  - This yields a state process that is continuous-time and discrete-state
- **In more complicated cases,**
  - the state process is e.g. a vector of integers (cf. loss and queueing network models)
- Typically we are interested in
  - whether the state process has a stationary distribution
  - if so, what it is?
- Although the state of the system did not follow the stationary distribution at time $0$, in many cases state distribution approaches the stationary distribution as $t$ tends to $\infty$.

### Bernoulli process

- **Definition:** Bernoulli process with success probability $p$ is an infinite series $(X_n \mid n = 1,2,...)$ of independent and identical random experiments of Bernoulli type with success probability $p$.
- Bernoulli process is clearly discrete-time and discrete-state
  - Parameter space: $I = \{1,2,\ldots\}$
  - State space: $S = \{0,1\}$
- Finite dimensional distributions (note: $X_n$’s are IID):
  $$P(X_1 = x_1, \ldots, X_n = x_n) = P(X_1 = x_1) \cdots P(X_n = x_n)$$
  $$= \prod_{i=1}^{n} p^{x_i} (1 - p)^{1-x_i} = p^{\sum_i x_i} (1 - p)^{n-\sum_i x_i}$$
- Bernoulli process is stationary (stationary distribution: Bernoulli($p$))

### Definition of a Poisson process

- Poisson process is the continuous-time counterpart of a Bernoulli process
  - It is a point process $(\tau_n \mid n = 1,2,...)$ where $\tau_n$ tells the occurrence time of the $n^{th}$ event, (e.g. arrival of a client)
  - “failure” in Bernoulli process is now an arrival of a client
- **Definition 1:** A point process $(\tau_n \mid n = 1,2,...)$ is a Poisson process with intensity $\lambda$, if the probability that there is an event during a short time interval $(t, t+h]$ is $\lambda h + o(h)$ independently of the other time intervals
  - $o(h)$ refers to any function such that $o(h)/h \to 0$ as $h \to 0$
  - new events happen with a constant intensity $\lambda$: $(\lambda h + o(h))/h \to \lambda$.
  - probability that there are no arrivals in $(t, t+h]$ is $1 - \lambda h + o(h)$
- Defined as a point process, Poisson process is discrete-time and continuous-state
  - Parameter space: $I = \{1,2,\ldots\}$
  - State space: $S = (0, \infty)$
5. Stochastic processes (1)

**Poisson process, another definition**

- Consider the interarrival time $\tau_n - \tau_{n-1}$ between two events ($\tau_0 = 0$)
  - Since the intensity that something happens remains constant $\lambda$, the ending of the interarrival time within a short period of time $(t, t+h)$, after it has lasted already the time $t$, does not depend on $t$ (or on other previous arrivals)
  - Thus, the interarrival times are independent and, additionally, they have the memoryless property. This property can be only the one of exponential distribution (of continuous-time distributions)
- **Definition 2**: A point process $(\tau_n | n = 1, 2, \ldots)$ is a **Poisson process** with intensity $\lambda$ if the interarrival times $\tau_n - \tau_{n-1}$ are independent and identically distributed (IID) with joint distribution $\text{Exp}(\lambda)$

5. Stochastic processes (1)

**Poisson process, yet another definition (1)**

- Consider finally the number of events $A(t)$ during time interval $[0, t]$
  - In a Bernoulli process, the number of successes in a fixed interval would follow a binomial distribution. As the "time slice" tends to 0, this approaches a Poisson distribution.
  - Note that $A(0) = 0$
- **Definition 3**: A counter process $(A(t) | t \geq 0)$ is a **Poisson process** with intensity $\lambda$ if its increments in disjoint intervals are independent and follow a Poisson distribution as follows:

$$A(t + \Delta) - A(t) \sim \text{Poisson}(\lambda \Delta)$$

- Defined as a counter process, Poisson process is continuous-time and discrete-state
  - Parameter space: $I = [0, \infty)$
  - State space: $S = \{0, 1, 2, \ldots\}$

5. Stochastic processes (1)

**Poisson process, yet another definition (2)**

- One dimensional distribution: $A(t) \sim \text{Poisson}(\lambda t)$
  - $E[A(t)] = \lambda t, \text{Var}[A(t)] = \lambda t$
- Finite dimensional distributions (due to independence of disjoint intervals):
  \[
P\{A(t_1) = x_1, \ldots, A(t_n) = x_n\} = \\
P\{A(t_1) = x_1\}P\{A(t_2) - A(t_1) = x_2 - x_1\} \cdots \\
P\{A(t_n) - A(t_{n-1}) = x_n - x_{n-1}\}
\]
- Poisson process, defined as a counter process is not stationary, but it has stationary increments
  - thus, it doesn‘t have a stationary distribution, but independent and identically distributed increments

5. Stochastic processes (1)

**Three ways to characterize the Poisson process**

- It is possible to show that all three definitions for a Poisson process are, indeed, equivalent
5. Stochastic processes (1)

Properties (1)

- **Property 1 (Sum):** Let $A_1(t)$ and $A_2(t)$ be two independent Poisson processes with intensities $\lambda_1$ and $\lambda_2$. Then the sum (superposition) process $A_1(t) + A_2(t)$ is a Poisson process with intensity $\lambda_1 + \lambda_2$.

- **Proof:** Consider a short time interval $(t, t+h]$
  - Probability that there are no events in the superposition is
  \[
  (1 - \lambda_1 h + o(h))(1 - \lambda_2 h + o(h)) = 1 - (\lambda_1 + \lambda_2)h + o(h)
  \]
  - On the other hand, the probability that there is exactly one event is
  \[
  (\lambda_1 h + o(h))(1 - \lambda_2 h + o(h)) + (1 - \lambda_1 h + o(h))(\lambda_2 h + o(h)) = (\lambda_1 + \lambda_2)h + o(h)
  \]

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Properties (2)

- **Property 2 (Random sampling):** Let $\tau_n$ be a Poisson process with intensity $\lambda$. Denote by $\sigma_n^{(1)}$ the point process resulting from a random and independent sampling (with probability $p$) of the points of $\tau_n$. Then $\sigma_n^{(1)}$ is a Poisson process with intensity $p\lambda$.

- **Proof:** Consider a short time interval $(t, t+h]$
  - Probability that there are no events after the random sampling is
  \[
  (1 - \lambda h + o(h)) + (1 - p)(\lambda h + o(h)) = 1 - p\lambda h + o(h)
  \]
  - On the other hand, the probability that there is exactly one event is
  \[
  p(\lambda h + o(h)) = p\lambda h + o(h)
  \]

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Properties (3)

- **Property 3 (Random sorting):** Let $\tau_n$ be a Poisson process with intensity $\lambda$. Denote by $\sigma_n^{(1)}$ the point process resulting from a random and independent sampling (with probability $p$) of the points of $\tau_n$. Denote by $\sigma_n^{(2)}$ the point process resulting from the remaining points. Then $\sigma_n^{(1)}$ and $\sigma_n^{(2)}$ are independent Poisson processes with intensities $\lambda p$ and $\lambda(1-p)$.

- **Proof:** Due to property 2, it is enough to prove that the resulting two processes are independent. Proof will be ignored on this course.

5. Stochastic processes (1)

Properties (4)

- **Property 4 (PASTA):** Consider any simple (and stable) teletraffic model with Poisson arrivals. Let $X(t)$ denote the state of system at time $t$ (continuous-time process) and $Y_n$ denote the state of the system seen by the $n$th arriving customer (discrete-time process). Then the steady-state distribution of $X(t)$ is the same as the stationary distribution of $Y_n$.

- **Proof:** Due to property 2, it is enough to prove that the resulting two processes are independent. Proof will be ignored on this course.

- **PASTA property is only valid for Poisson arrivals**

  - and it is not valid for other arrival processes

  - consider e.g. your own PC. Whenever you start a new session, the system is idle. In the continuous time, however, the system is not only idle but also busy (when you use it).
Example (1)

- Connection requests arrive at a server according to a Poisson process with intensity \( \lambda = 5 \) requests in a minute.
  - What is the probability that exactly 2 new requests arrive during the next 30 seconds?
    - Number of new arrivals during a time interval follows Poisson distribution with the parameter \( \lambda \Delta = 5 / 60 \cdot 30 = 2.5 \)
  
  \[
  A(t + 30) - A(t) \sim \text{Poisson}(2.5)
  \]

  \[P(A(t + 30) - A(t) = 2) = \frac{2.5^2}{2!} e^{-2.5} = 0.257\]

Example (2)

- Consider the system described on previous slide.
  - A new connection request has just arrived at the server. What is the probability that it takes more than 30 seconds before next request arrives?
    - Consider the process as a point process. The interarrival time follows exponential distribution with parameter \( \lambda \).

  \[
P(\tau_{i+1} - \tau_i \geq 30) = 1 - P(\tau_{i+1} - \tau_i \leq 30) = e^{-\lambda / 60} e^{-30} = e^{-2.5} = 0.082
  \]

- Consider the process as a counter process, cf. slide 29. Now we can restate the question above as "What is the probability that there are no arrivals during 30 seconds?"

  \[
P(A(t + 30) - A(t) = 0) = \frac{2.5^0}{0!} e^{-2.5} = e^{-2.5} = 0.082
  \]