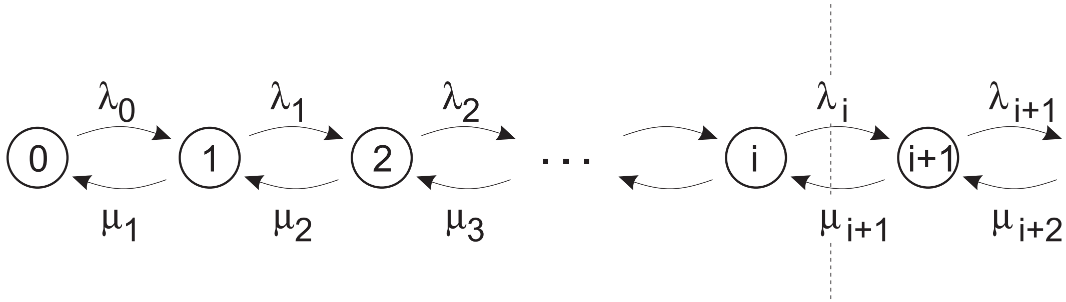
Birth-death processes

General

A birth-death (BD process) process refers to a Markov process with

- a discrete state space
- the states of which can be enumerated with index i=0,1,2,...such that
- state transitions can occur only between neighbouring states, $i \to i+1$ or $i \to i-1$



Transition rates

$$q_{i,j} = \begin{cases} \lambda_i & \text{when} \quad j = i+1 \\ \mu_i & \text{if} \quad j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

probability of death in interval Δt on $\lambda_i \Delta t$
probability of birth in interval Δt on $\mu_i \Delta t$
when the system is in state i

The equilibrium probabilities of a BD process

We use the method of a cut = global balance condition applied on the set of states $0, 1, \ldots, k$.

In equilibrium the probability flows across the cut are balanced (net flow =0)

$$\boxed{\lambda_k \pi_k = \mu_{k+1} \pi_{k+1}} \qquad k = 0, 1, 2, \ldots$$

We obtain the recursion

$$\pi_{k+1} = \frac{\lambda_k}{\mu_{k+1}} \pi_k$$

By means of the recursion, all the state probabilities can be expressed in terms of that of the state 0, π_0,

$$\boxed{\pi_k = \frac{\lambda_{k-1}\lambda_{k-1} \cdots \lambda_0}{\mu_k \mu_{k-1} \cdots \mu_1} \pi_0 = \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}} \pi_0}$$

The probability π_0 is determined by the normalization condition π_0

$$\boxed{\pi_0 = \frac{1}{1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \cdots} = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}}}$$

The time-dependent solution of a BD process

Above we considered the equilibrium distribution $\boldsymbol{\pi}$ of a BD process.

Sometimes the state probabilities at time 0, $\boldsymbol{\pi}(0)$, are known

- usually one knows that the system at time 0 is precisely in a given state k; then $\pi_k(0) = 1$ and $\pi_j(0) = 0$ when $j \neq k$

and one wishes to determine how the state probabilities evolve as a function of time $\boldsymbol{\pi}(t)$

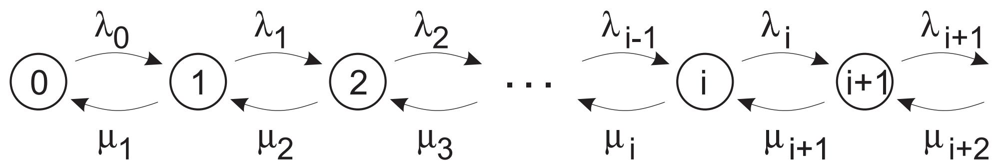
- in the limit we have $\lim_{t\to\infty} \boldsymbol{\pi}(t) = \boldsymbol{\pi}$.

This is determined by the equation

$$\boxed{\frac{d}{dt}\boldsymbol{\pi}(t) = \boldsymbol{\pi}(t) \cdot \mathbf{Q}} \qquad \text{where}$$

$$\mathbf{Q} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \dots & \dots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 \\ \vdots & 0 & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 \\ \vdots & \vdots & 0 & \mu_4 & -(\lambda_4+\mu_4) \end{pmatrix}$$

The time-dependent solution of a BD process (continued)



The equations component wise

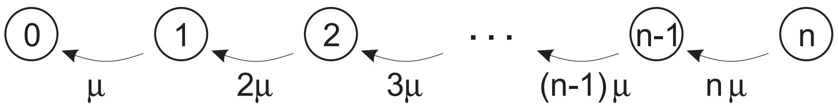
$$\begin{cases} \dfrac{d\pi_i(t)}{dt} = \underbrace{-(\lambda_i+\mu_i)\pi_i(t)}_{\text{flows out}} + \underbrace{\lambda_{i-1}\pi_{i-1}(t)+\mu_{i+1}\pi_{i+1}(t)}_{\text{flows in}} \quad i=1,2,\ldots \\ \dfrac{d\pi_0(t)}{dt} = \underbrace{-\lambda_0\pi_0(t)}_{\text{flow out}} + \underbrace{\mu_1\pi_1(t)}_{\text{flow in}} \end{cases}$$

Example 1. Pure death process

$$\begin{cases} \lambda_i = 0 \\ \mu_i = i\mu \end{cases} \quad i = 0, 1, 2, \ldots \qquad\qquad \pi_i(0) = \begin{cases} 1 & i = n \\ 0 & i \neq n \end{cases}$$

all individuals have the same mortality rate μ

the system starts from state n



State 0 is an absorbing state, other states are transient

$$\begin{cases} \frac{d}{dt}\pi_n(t) &= -n\mu\pi_n(t) & \Rightarrow \quad \pi_n(t) = e^{-n\mu t} \\ \frac{d}{dt}\pi_i(t) &= (i+1)\mu\pi_{i+1}(t) - i\mu\pi_i(t) & i = 0, 1, \ldots, n-1 \end{cases}$$

$$\frac{d}{dt}(e^{i\mu t}\pi_i(t)) = (i+1)\mu\pi_{i+1}(t)e^{i\mu t} \quad \Rightarrow \quad \pi_i(t) = (i+1)e^{-i\mu t}\mu\int_0^t \pi_{i+1}(t')e^{i\mu t'}dt'$$

$$\pi_{n-1}(t) = ne^{-(n-1)\mu t}\mu\int_0^t \underbrace{e^{-n\mu t'}e^{(n-1)\mu t'}}_{e^{-\mu t'}}dt' = n\,e^{-(n-1)\mu t}(1 - e^{-\mu t})$$

Recursively $\boxed{\pi_i(t) = \binom{n}{i}(e^{-\mu t})^i(1 - e^{-\mu t})^{n-i}}$

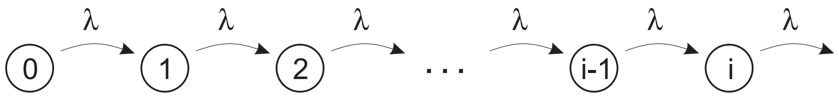
Binomial distribution: the survival probability at time t is $e^{-\mu t}$ independent of others

Example 2. Pure birth process (Poisson process)

$$\begin{cases} \lambda_i = \lambda \\ \mu_i = 0 \end{cases} \quad i = 0, 1, 2, \ldots \qquad\qquad \pi_i(0) = \begin{cases} 1 & i = 0 \\ 0 & i > 0 \end{cases}$$

birth probability per time unit is constant λ

initially the population size is 0



All states are transient

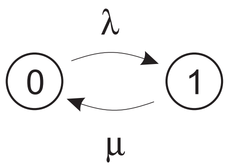
$$\begin{cases} \frac{d}{dt}\,\pi_i(t) \;=\; -\lambda\pi_i(t) + \lambda\pi_{i-1}(t) & i > 0 \\ \frac{d}{dt}\,\pi_0(t) \;=\; -\lambda\pi_0(t) & \Rightarrow \quad \pi_0(t) = e^{-\lambda t} \end{cases}$$

$$\frac{d}{dt}(e^{\lambda t}\pi_i(t)) \;=\; \lambda\pi_{i-1}(t)e^{\lambda t} \qquad \Rightarrow \qquad \pi_i(t) \;=\; e^{-\lambda t}\lambda \int_0^t \pi_{i-1}(t')e^{\lambda t'}dt'$$

$$\pi_1(t) \;=\; e^{-\lambda t}\lambda \int_0^t \underbrace{e^{-\lambda t'}e^{\lambda t'}}_{1} dt' \;=\; e^{-\lambda t}(\lambda t)$$

Recursively $\boxed{\pi_i(t) = \frac{(\lambda t)^i}{i!}e^{-\lambda t}}$ Number of births in interval $(0, t) \sim \text{Poisson}(\lambda t)$

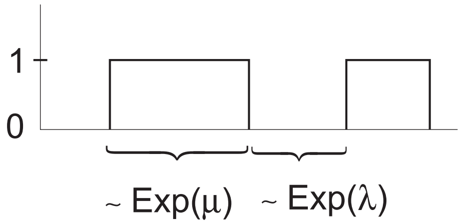
Example 3. A single server system



- constant arrival rate λ (Poisson arrivals)
- stopping rate of the service μ (exponential distribution)

The states of the system

$$\begin{cases} 0 & \text{server free} \\ 1 & \text{server busy} \end{cases}$$



$$\begin{cases} \frac{d}{dt}\pi_0(t) = -\lambda\pi_0(t) + \mu\pi_1(t) \\ \frac{d}{dt}\pi_1(t) = \lambda\pi_0(t) - \mu\pi_1(t) \end{cases} \qquad \mathbf{Q} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

BY adding both sides of the equations

$$\frac{d}{dt}(\pi_0(t) + \pi_1(t)) = 0 \quad \Rightarrow \quad \pi_0(t) + \pi_1(t) = \text{constant} = 1 \quad \Rightarrow \quad \pi_1(t) = 1 - \pi_0(t)$$

$$\frac{d}{dt}\pi_0(t) + (\lambda+\mu)\pi_0(t) = \mu \quad \Rightarrow \quad \frac{d}{dt}(e^{(\lambda+\mu)t}\pi_0(t)) = \mu e^{(\lambda+\mu)t}$$

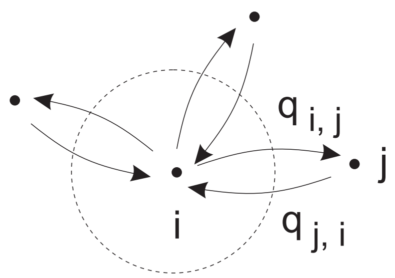
$$\boxed{\begin{aligned} \pi_0(t) &= \tfrac{\mu}{\lambda+\mu} + (\pi_0(0) - \tfrac{\mu}{\lambda+\mu})e^{-(\lambda+\mu)t} \\ \pi_1(t) &= \tfrac{\lambda}{\lambda+\mu} + (\pi_1(0) - \tfrac{\lambda}{\lambda+\mu})e^{-(\lambda+\mu)t} \end{aligned}}$$

$$\underbrace{\phantom{\tfrac{\lambda}{\lambda+\mu}}}_{\substack{\text{equilibrium} \\ \text{distribution}}} \quad \underbrace{\phantom{(\pi_1(0) - \tfrac{\lambda}{\lambda+\mu})}}_{\substack{\text{deviation from} \\ \text{the equilibrium}}} \quad \underbrace{\phantom{e^{-(\lambda+\mu)t}}}_{\substack{\text{decays expo-} \\ \text{nentially}}}$$

Summary of the analysis on Markov processes

1. Find the state description of the system
   - no ready recipe
   - often an appropriate description is obvious
   - sometimes requires more thinking
   - a system may Markovian with respect to one state description but non-Markovian with respect to another one (the state information is not sufficient)
   - finding the state description is the creative part of the problem
2. Determine the state transition rates
   - a straight forward task when holding times and interarrival times are exponential
3. Solve the balance equations
   - in principle straight forward (solution of a set of linear equations)
   - the number of unknowns (number of states) can be very great
   - often the special structure of the transition diagram can be exploited

Global balance



$n+1$ states

$$\underbrace{\boxed{\sum_{j \neq i} \pi_j q_{j,i}}}_{\text{flow to state } i} = \underbrace{\sum_{j \neq i} \pi_i q_{i,j}}_{\text{flow out of state } i}$$

$i = 0, 1, \ldots, n$
one equation per each state

$$\overbrace{(\pi_0, \ldots, \pi_n)}^{\boldsymbol{\pi}} \overbrace{\begin{pmatrix} -\sum\limits_j q_{0,j} & q_{0,1} & q_{0,2} & \cdots & q_{0,n} \\ q_{1,0} & -\sum\limits_j q_{1,j} & q_{1,2} & \cdots & q_{1,n} \\ q_{2,0} & q_{2,1} & -\sum\limits_j q_{2,j} & \cdots & q_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{n,0} & q_{n,1}, & q_{n,2}, & \cdots & -\sum\limits_j q_{n,j} \end{pmatrix}}^{\mathbf{Q}} = \begin{pmatrix} 0 \\ 0 \\ \\ \vdots \\ 0 \end{pmatrix}$$

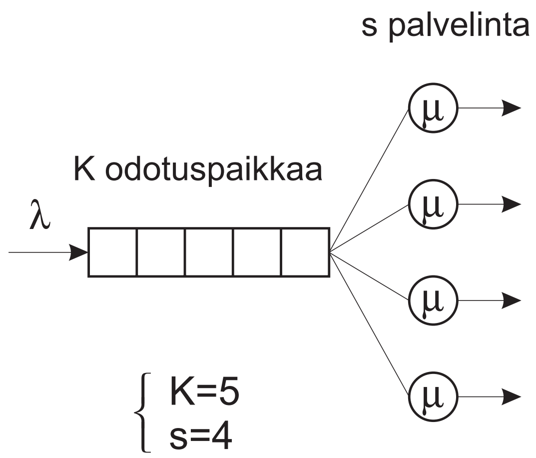
$$\boxed{\boldsymbol{\pi} \cdot \mathbf{Q} = \mathbf{0}}$$

one equation is redundant

$$\boxed{\pi_0 + \pi_1 + \cdots + \pi_n = 1}$$

normalization condition

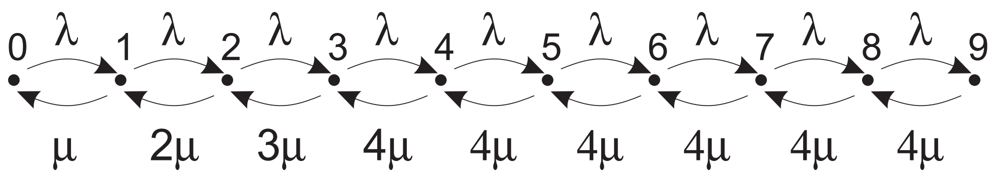
Example 1. A queueing system



$$\begin{cases} s & \text{servers} \\ K & \text{waiting places} \\ \lambda & \text{arrival rate (Poisson)} \\ \mu & \text{Exp}(\mu) \text{ holding time (expectation } 1/\mu) \end{cases}$$

The number of customers in system N is an appropriate state variable

- uniquely determines the number of customers in service and in waiting room
- after each arrival and departure the remaining service times of the customers in service are $\text{Exp}(\mu)$ distributed (memoryless)

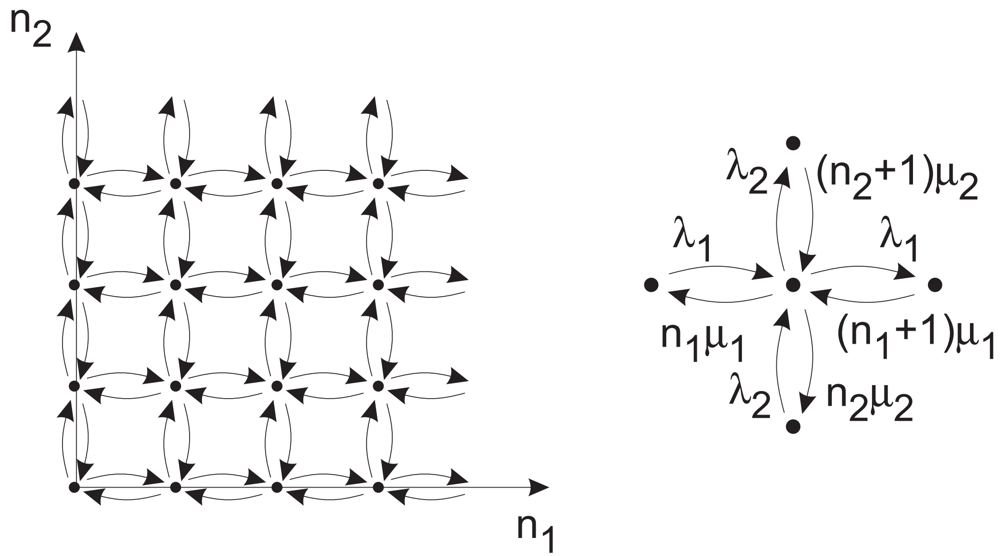


Example 2. Call blocking in an ATM network

A virtual path (VP) of an ATM network is offered calls of two different types.

$$\begin{cases} R_1 &= \text{1Mbps} \\ \lambda_1 &= \text{arrival rate} \\ \mu_1 &= \text{mean holding time} \end{cases} \qquad \begin{cases} R_2 &= \text{2Mbps} \\ \lambda_2 &= \text{arrival rate} \\ \mu_2 &= \text{mean holding time} \end{cases}$$

a) The capacity of the link is large (infinite)



Call blocking in an ATM network (continued)

b) The capacity of the link is 4.5 Mbps