Tandem fluid queues fed by homogeneous on-off sources

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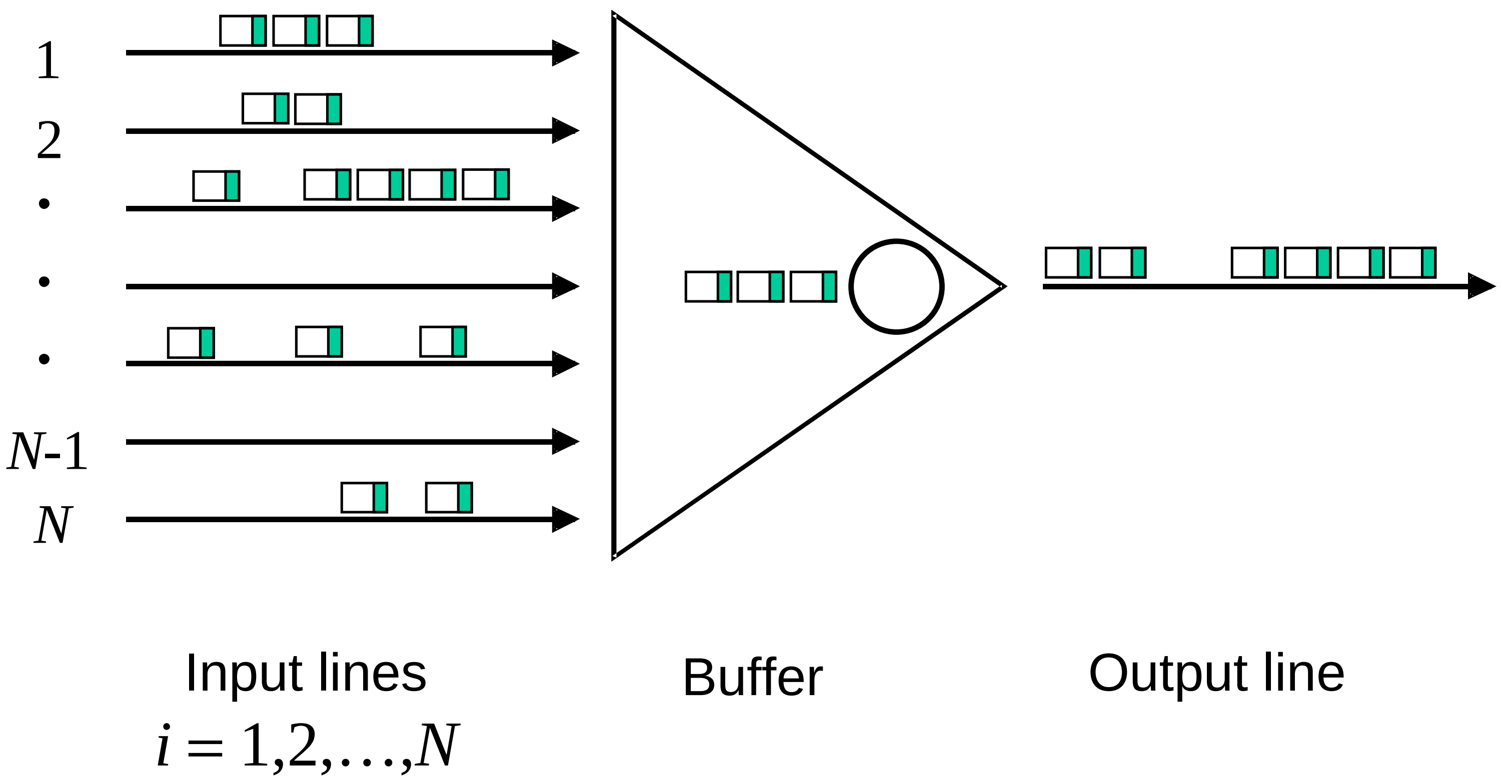
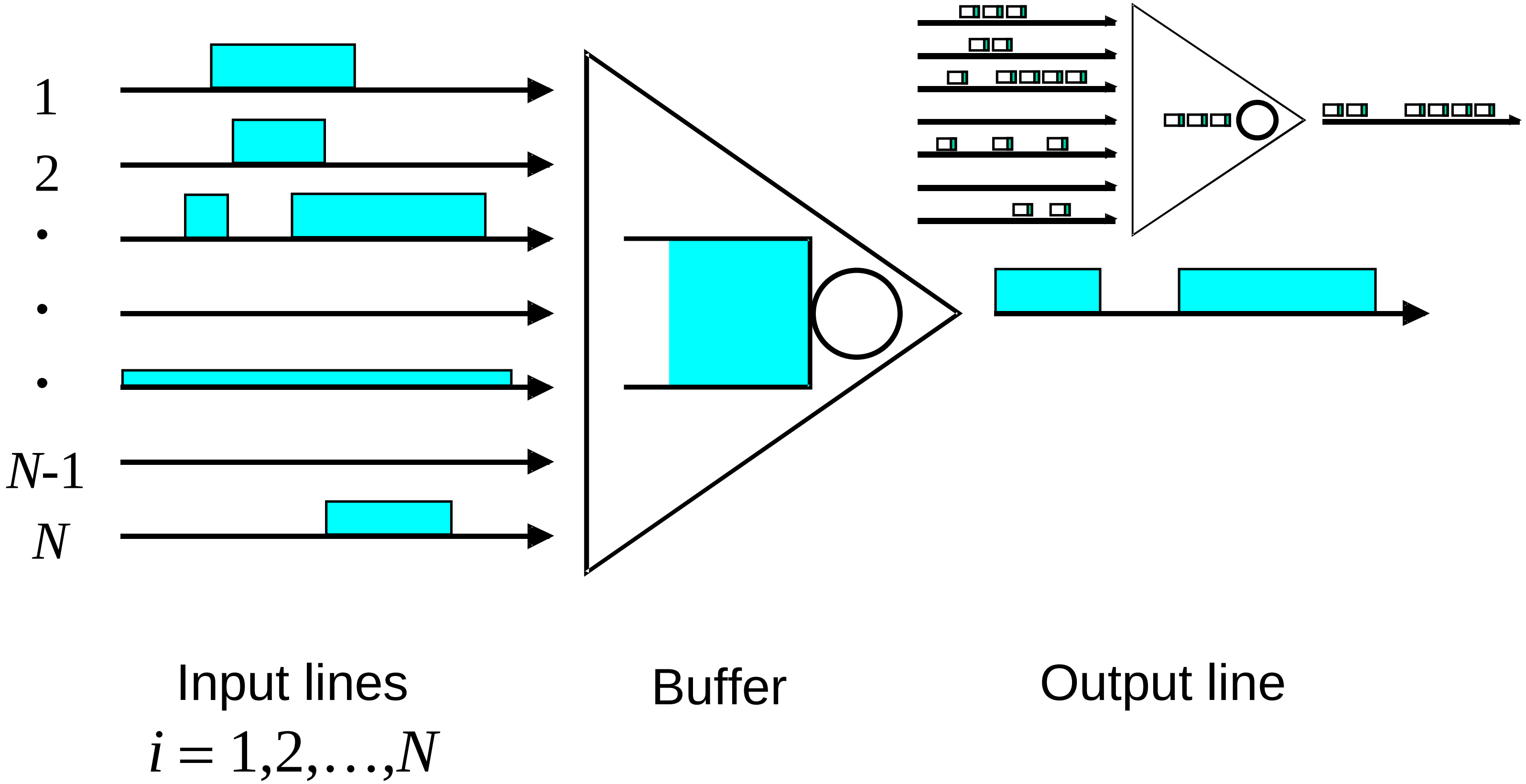
Contents

- Introduction
- Known results about fluid queues
- New results about tandem fluid queues

Statistical multiplexer ...

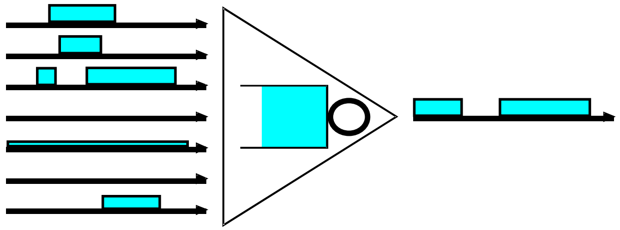


... as seen at the burst level ...



… = Fluid queue

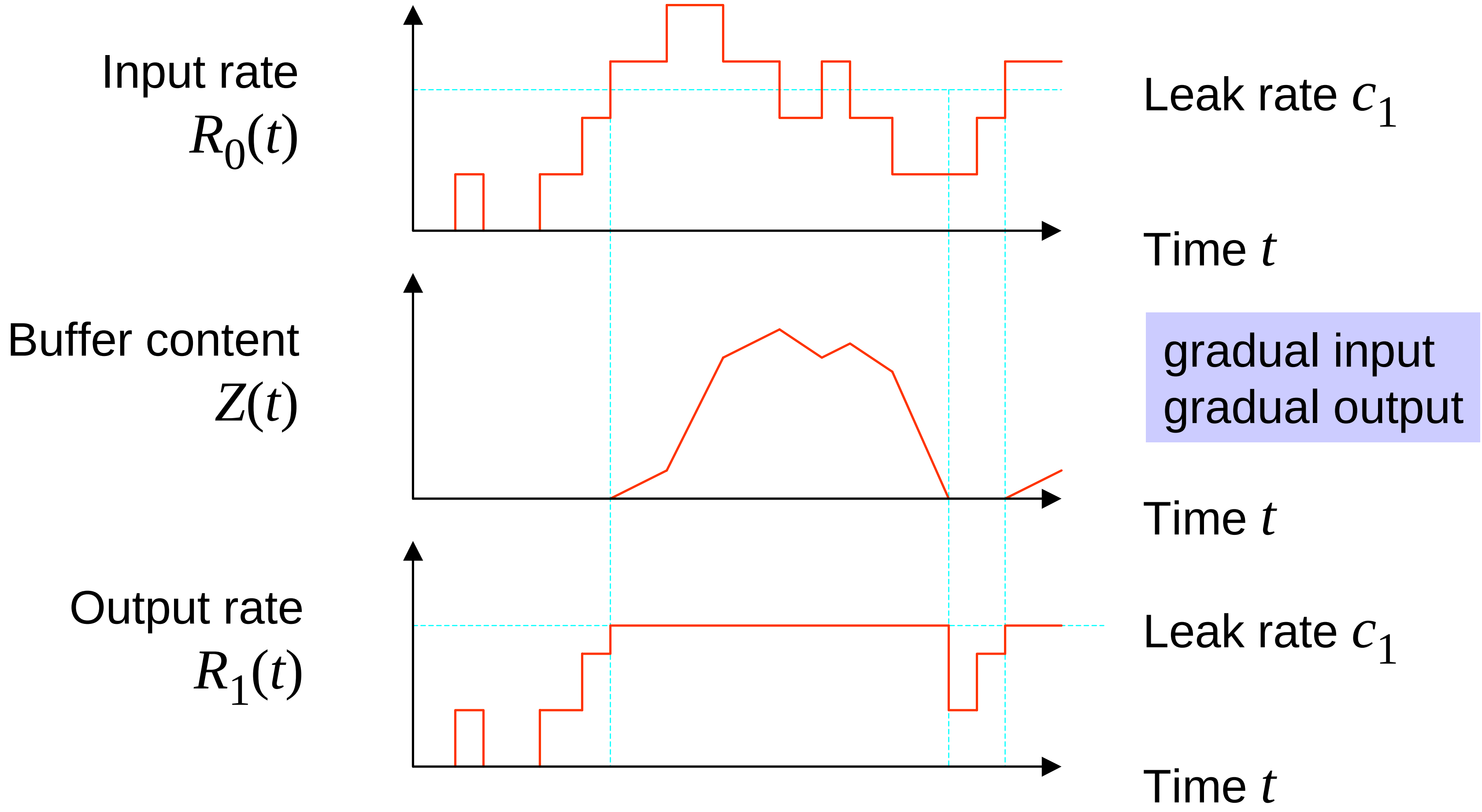
- Input rate $R_0(t)$
  - varying randomly
  - gradual input!
- Buffer size
  - we assume: infinite
- Leak rate c_1
  - max output rate
  - gradual output!
- Buffer content $Z(t)$
- Output rate $R_1(t)$



$$Z(t) = Z(0) + \int_0^t R_0(u)\,du - \int_0^t R_1(u)\,du$$

$$R_1(t) = \begin{cases} \min\{R_0(t), c_1\}, & \text{if } Z(t) = 0 \\ c_1, & \text{if } Z(t) > 0 \end{cases}$$

Evolution


Input rate
$R_0(t)$
Leak rate c_1
Time t
Buffer content
$Z(t)$
gradual input
gradual output
Time t
Output rate
$R_1(t)$
Leak rate c_1
Time t

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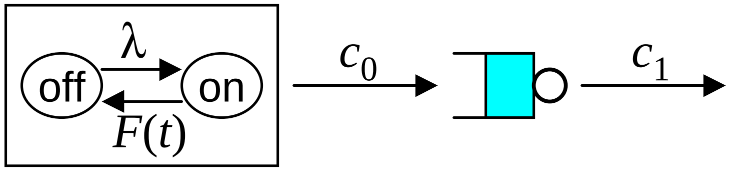
Fluid queue fed by a single on-off source

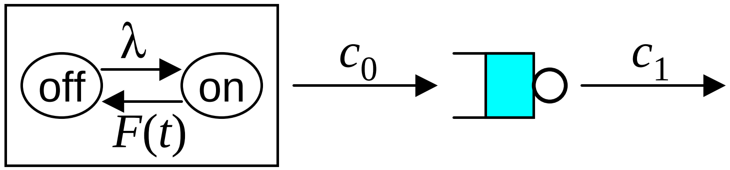
- **Input**: On-off source with rate c_0
  - silent periods $S_0 \sim \mathrm{Exp}(\lambda)$
  - active periods $A_0 \sim F(t)$
- Natural assumption:



$$c_0 > c_1$$

- **Buffer content**: $Z \sim$ **?**
- **Output**: looks like another on-off source with
  - silent periods $S_1 \sim \mathrm{Exp}(\lambda)$
  - active periods $A_1 \sim$ **?**

Elementary results

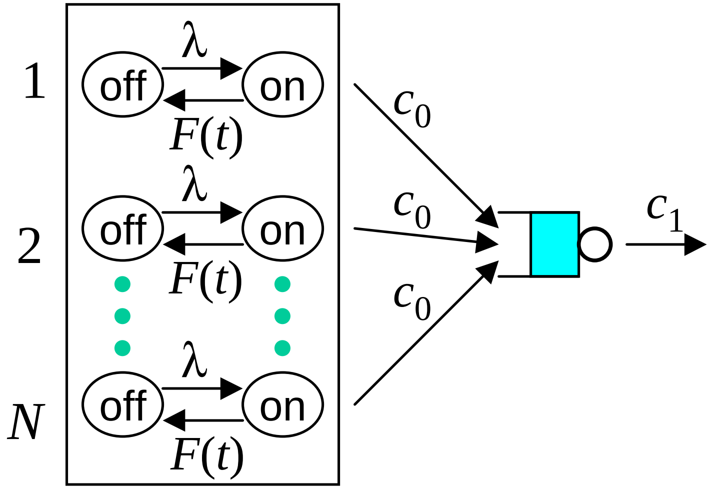


- **Buffer content**:
  - $P\{Z > z\} = \gamma P\{V > z\}$
  - where V ~ workload in a certain M/G/1 queue
- **Output**:
  - A_1 ~ busy period in another M/G/1 queue

Fluid queue fed by multiple on-off sources

- **Input**: On-off sources with rate c_0
  - silent periods $S_0 \sim \mathrm{Exp}(\lambda)$
  - active periods $A_0 \sim F(t)$
- Restrictive assumption:

$$c_0 \geq c_1$$



- **Output**: looks like another on-off source with
  - silent periods $S_1 \sim \mathrm{Exp}(N\lambda)$
  - active periods $A_1 \sim$ **?**
- Boxma & Dumas (1998), Aalto (1998):
  - $A_1 \sim$ busy period in a certain M/G/1 queue

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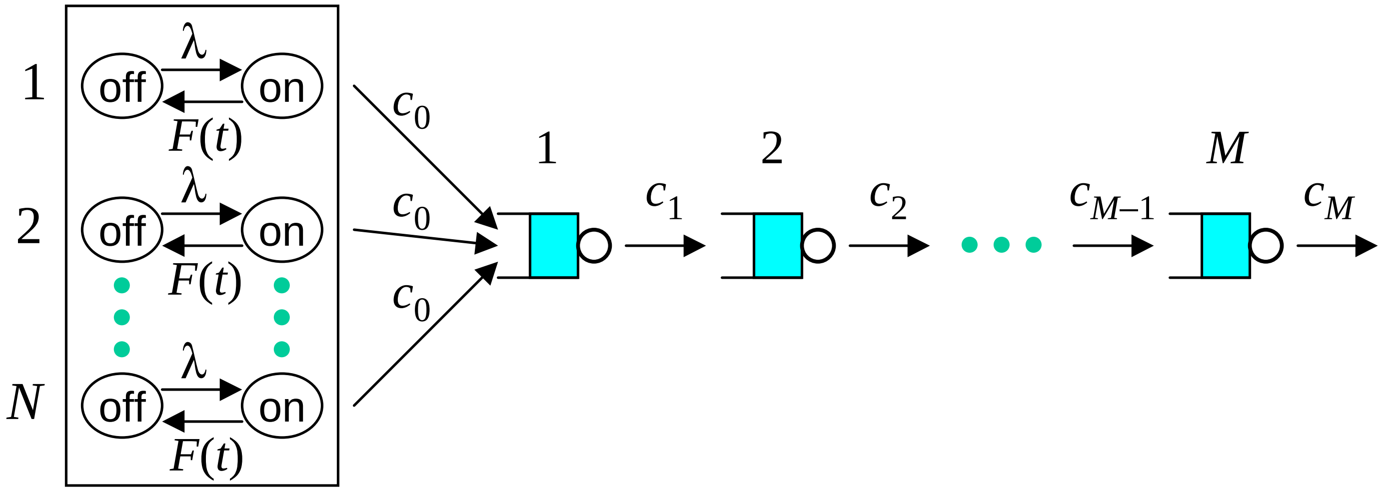
Tandem fluid queue fed by multiple on-off sources



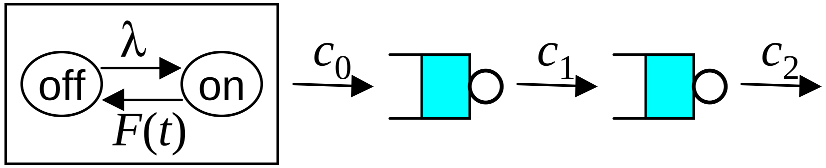
- Natural assumption:

$$Nc_0 > c_1 > c_2 > \ldots > c_M$$

- Observation:

$$Z_i(t) > 0 \;\Rightarrow\; Z_{i+k}(t) > 0$$

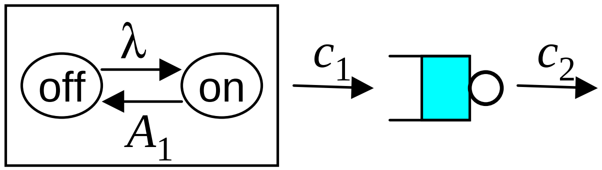
Tandem fluid queue fed by a single on-off source



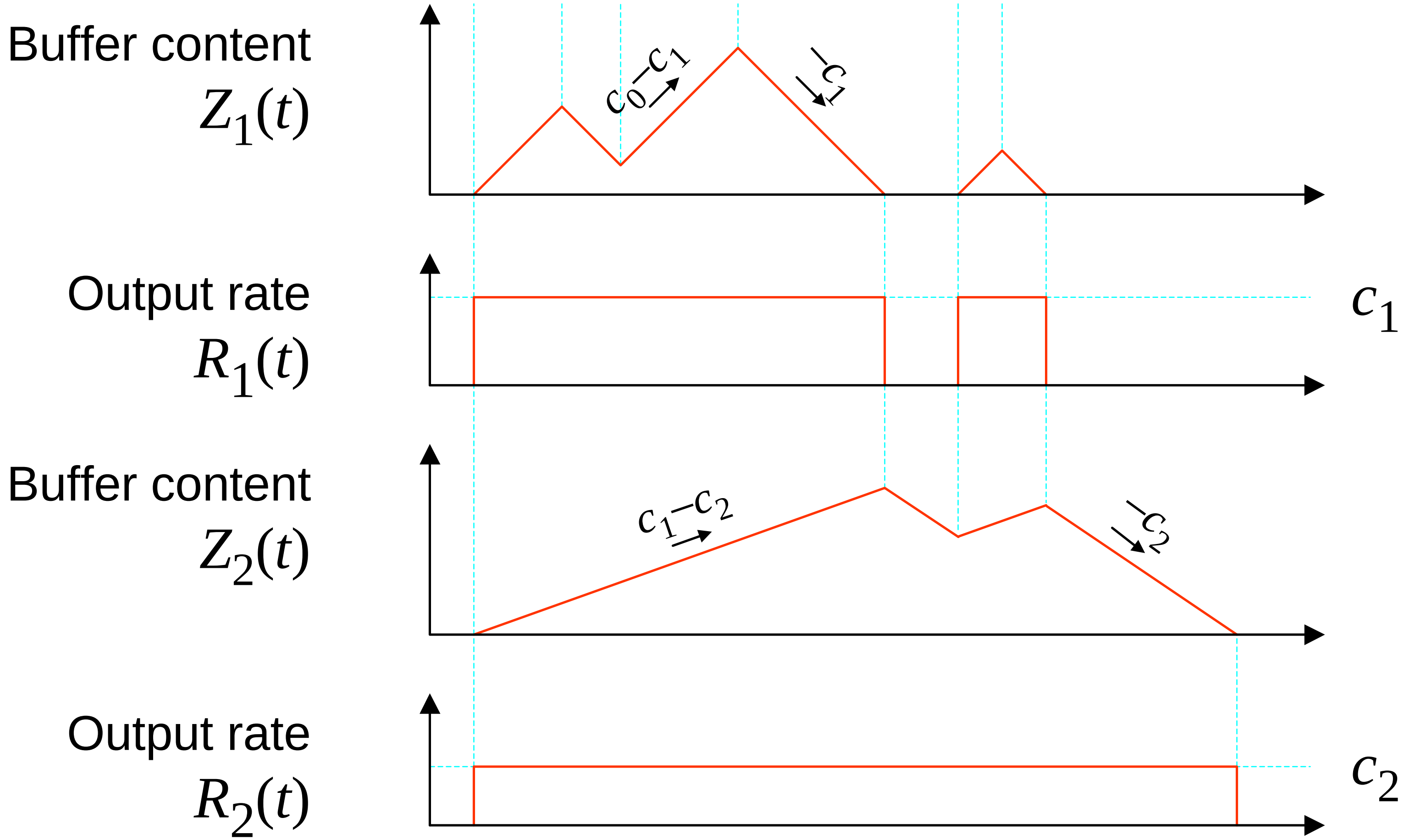
- Natural assumption:

$$c_0 > c_1 > c_2$$

- **Content of buffer** 2: $Z_2 \sim$ **?**
- **Output from buffer** 2: looks like another on-off source with
  - silent periods $S_2 \sim \mathrm{Exp}(\lambda)$
  - active periods $A_2 \sim$ **?**

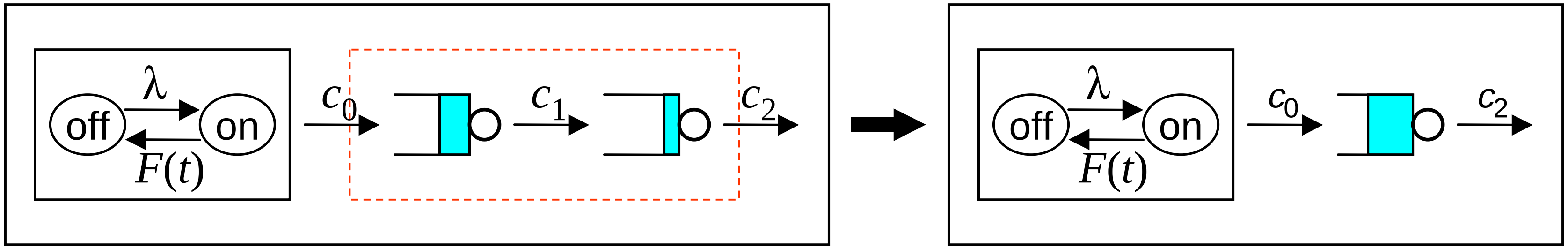


Evolution


Buffer content
$Z_1(t)$
$c_0 - c_1$
$-c_1$
Output rate
$R_1(t)$
c_1
Buffer content
$Z_2(t)$
$c_1 - c_2$
$-c_2$
Output rate
$R_2(t)$
c_2

Output from buffer *i*

- **New result**:
  - A_i ~ busy period in an M/G/1 queue with arrival rate $\lambda(c_0 - c_i)/c_0$ and service time d.f. $F(c_i t/c_0)$
- Idea of the proof for $i = 2$: combine the two buffers



$$\forall t : Z_1(t) + Z_2(t) = \tilde{Z}_1(t)$$

$\Rightarrow$ outputs from the two systems are identical!

Content of buffer *i*

- Let

$$\alpha_i(\theta) = E[e^{-\theta A_i}] \quad \rho_i = \frac{c_0}{c_i}\frac{\lambda\beta_1}{1+\lambda\beta_1} \quad \gamma_i = \frac{c_{i-1}(1-\rho_{i-1})}{c_{i-1}-c_i} \quad \kappa_i = \frac{\rho_i}{1-\rho_i}$$

- **New result**: If $\rho_i < 1$, then

$$E[e^{-\theta Z_i}] = 1-\gamma_i + \gamma_i \frac{(c_i - (c_{i-1}-c_i)\kappa_{i-1})\theta}{c_i\theta - \lambda + \lambda\alpha_{i-1}((c_{i-1}-c_i)\theta)}$$

- Note: This is an implicit equation for the LST
- For exponential active periods, the transform can be inverted!

Exponential active periods

- Let

$$\Theta_i = \frac{\lambda + (\mu - \lambda) c_{i-1} / c_0}{c_{i-1} - c_i} \quad \eta_i = \frac{\mu}{c_0 - c_i} - \frac{\lambda}{c_i} \quad \omega_i = \frac{4\lambda\mu c_{i-1}(c_0 - c_{i-1})}{c_0^2 (c_{i-1} - c_i)^2}$$

- **New result**: If $\rho_i < 1$, then

$$P\{Z_i \in dy\} = (1-\rho_i)\delta_0(y)dy + (1-\rho_i)e^{-\eta_i y} \times \left(\frac{\lambda c_{i-1}}{c_i(c_{i-1}-c_i)} - \frac{c_0 \omega_i}{2(c_0 - c_i)} \int_0^y e^{-(\Theta_i - \eta_i)u} \frac{I_1(u\sqrt{\omega_i})}{u\sqrt{\omega_i}} du \right) dy$$

  - where δ_0 denotes the Dirac measure at 0 and I_1 the modified Bessel function of the first kind of order 1

First moments

- Let

$$\beta_k = E[A_0^k] \qquad \tilde{\beta}_k = E[(c_0 A_0)^k]$$

- **New result**: If $\rho_i < 1$, then

$$E[Z_i] = \frac{\tilde{\beta}_2}{2\tilde{\beta}_1}\left(\frac{1}{1+\lambda\beta_1}\right)^2 (\kappa_i - \kappa_{i-1})$$

$$E[Z_i^2] = \frac{\tilde{\beta}_3}{3\tilde{\beta}_1}\left(\frac{1}{1+\lambda\beta_1}\right)^3 \frac{(\kappa_i - \kappa_{i-1})^2}{\kappa_i} +$$

$$2\left(\frac{\tilde{\beta}_2}{2\tilde{\beta}_1}\right)^2\left(\frac{1}{1+\lambda\beta_1}\right)^4 \frac{(\kappa_i - \kappa_{i-1})^2}{\kappa_i}(\kappa_{i-1} + \kappa_i - 2\lambda\beta_1)$$

Correlation between the buffer contents

- Let

$$b = \frac{2\beta_3\beta_1}{3\beta_2^2}(1+\lambda\beta_1) - 2\lambda\beta_1$$

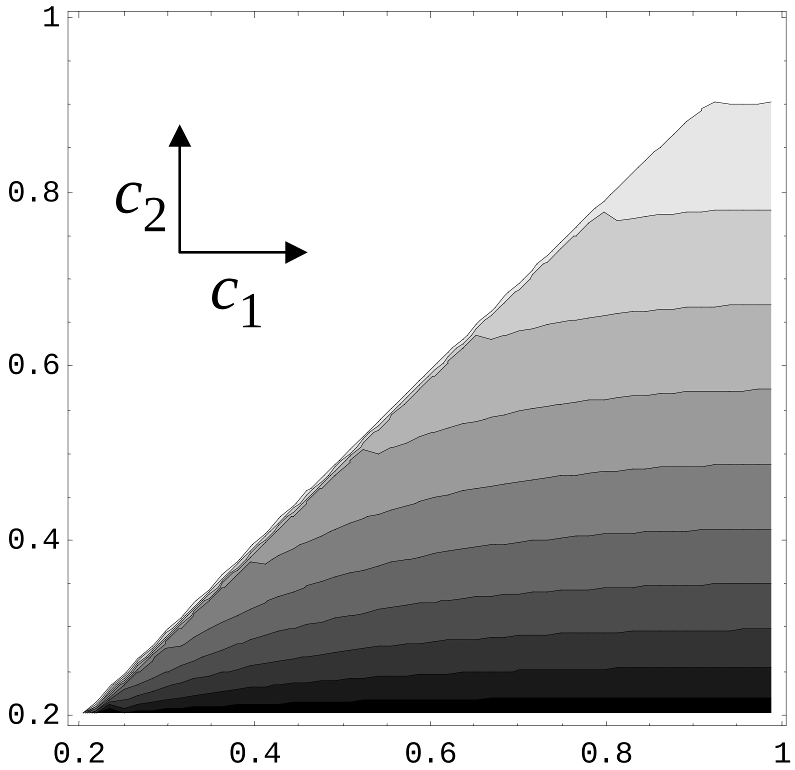
- **New result**: If $\rho_2 < 1$, then

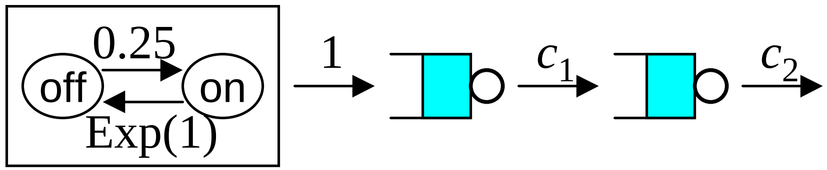
$$\mathrm{Corr}[Z_1, Z_2] = \frac{b(\kappa_0+\kappa_1)+\kappa_0^2+\kappa_0\kappa_1+\kappa_1^2}{\sqrt{(2b+2\kappa_0+\kappa_1)\kappa_1}\sqrt{(2b+2\kappa_1+\kappa_2)\kappa_2}} > 0$$

- Idea of the proof:

$$2\,\mathrm{Cov}[Z_1, Z_2] = \mathrm{Var}[Z_1+Z_2] - \mathrm{Var}[Z_1] - \mathrm{Var}[Z_2]$$

Example (1)

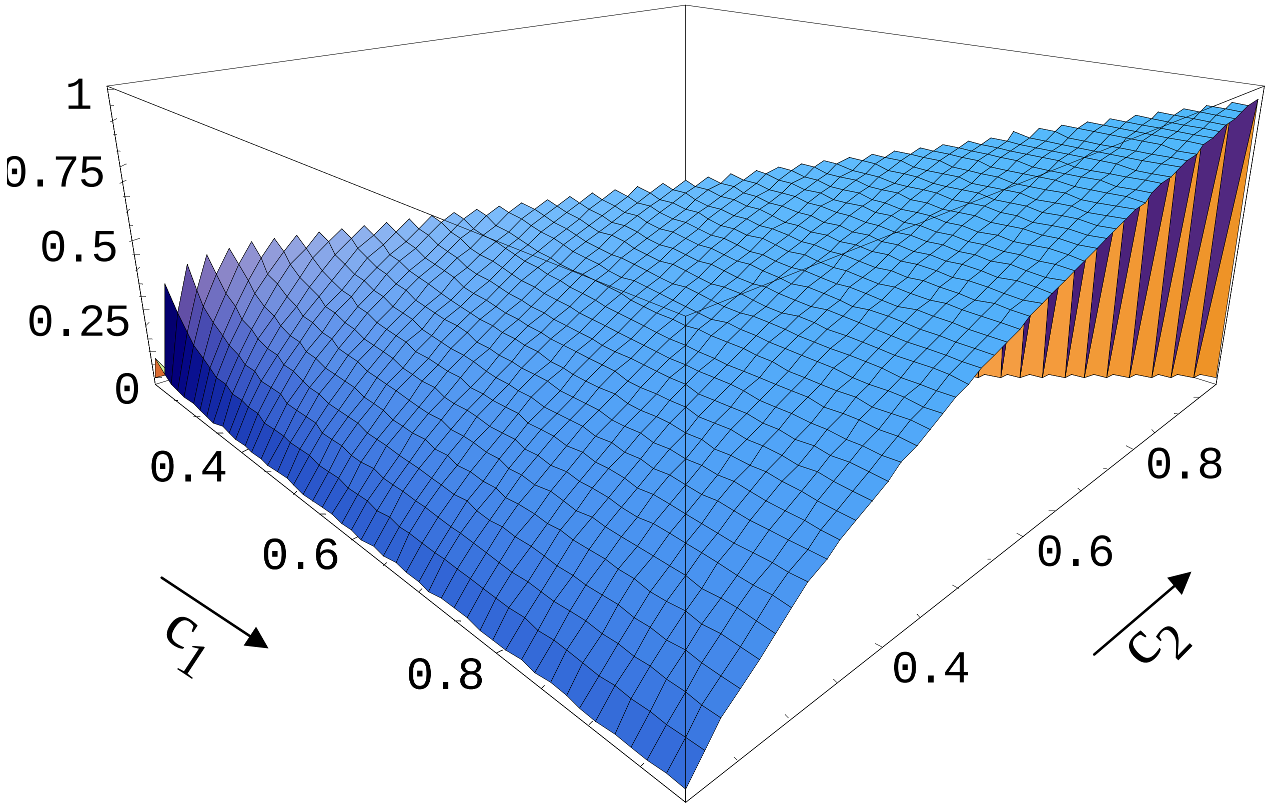




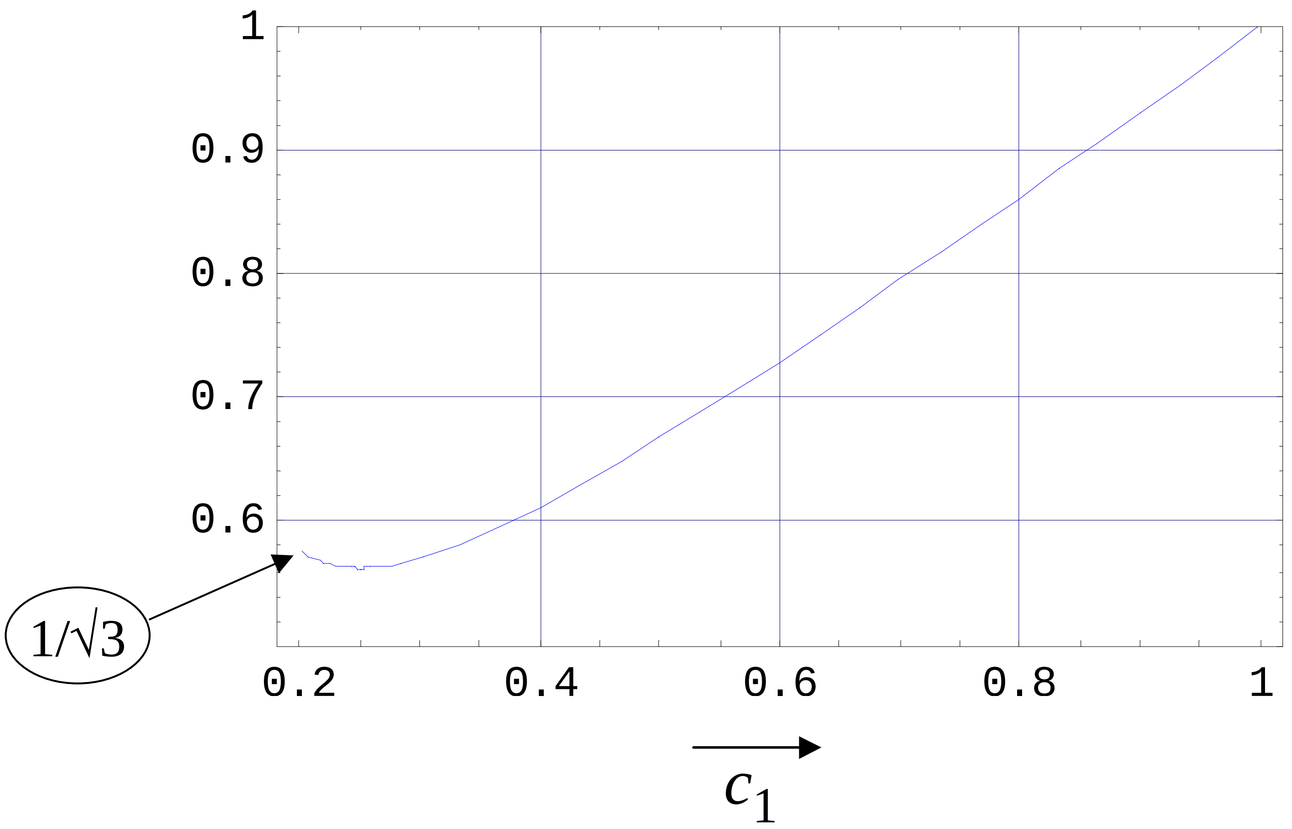
Stability condition:

$$1 = c_0 > c_1 > c_2 > 0.2$$

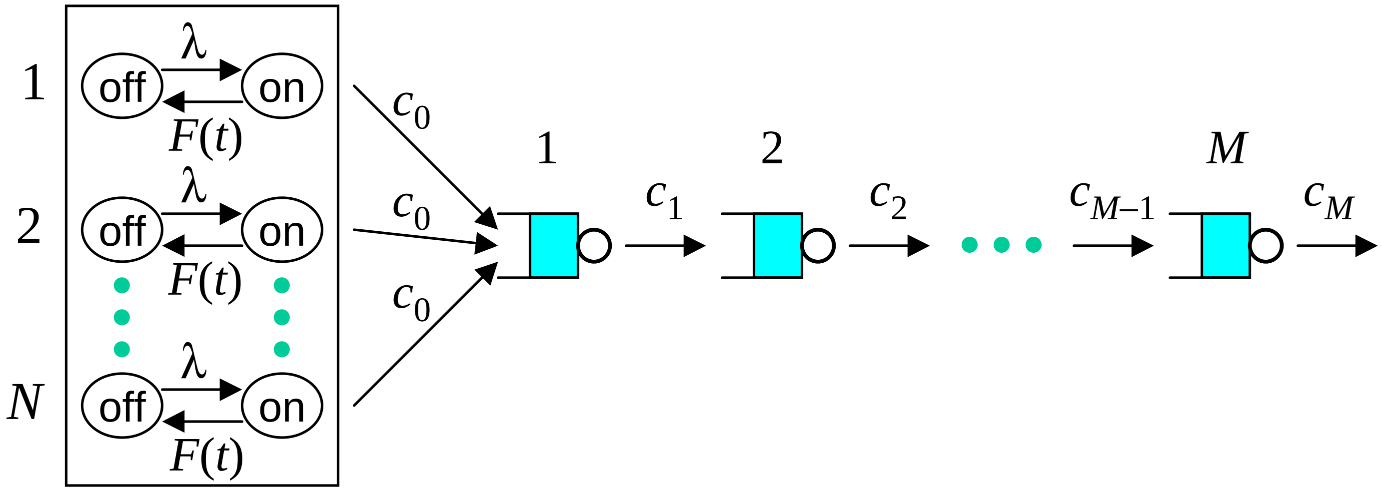
Example (2)



Example (3)



Tandem fluid queue fed by multiple on-off sources



- Restrictive assumption:

$$c_0 \geq c_1 > c_2 > \ldots > c_M$$

$\Rightarrow$ output from each buffer i looks like an on-off source

- Similar results as for the single source case available from the second buffer on!

THE END