Aalto University
School of Electrical
Engineering

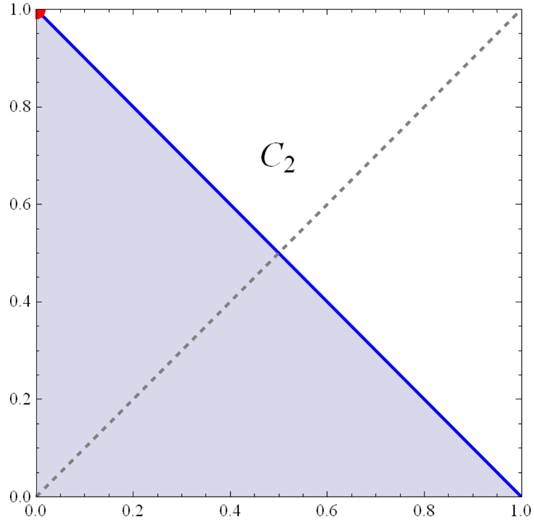
Optimal scheduling problem for scalable queues

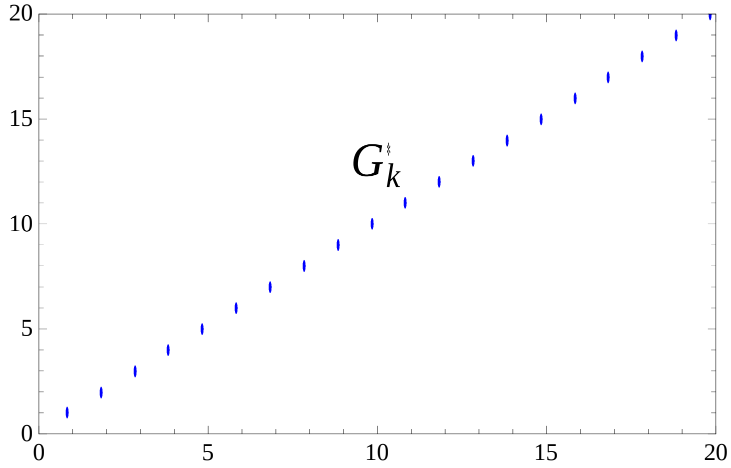
Samuli Aalto

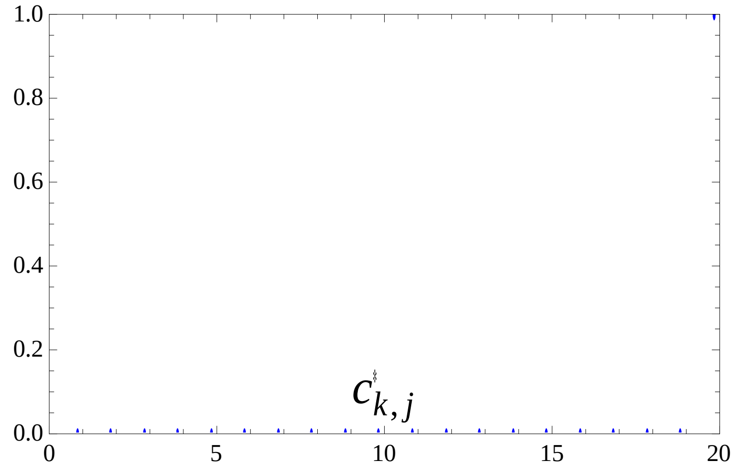
Aalto University, Finland

Dagstuhl Seminar 13111: Scheduling
10–15 March 2013
Wadern, Germany

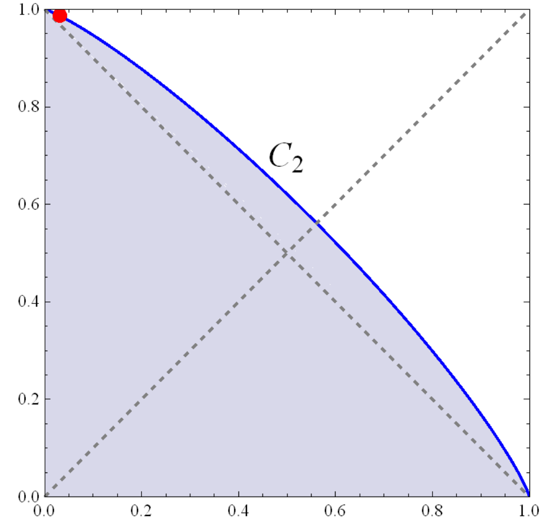
Alpha = 1.0
(single-server queue)

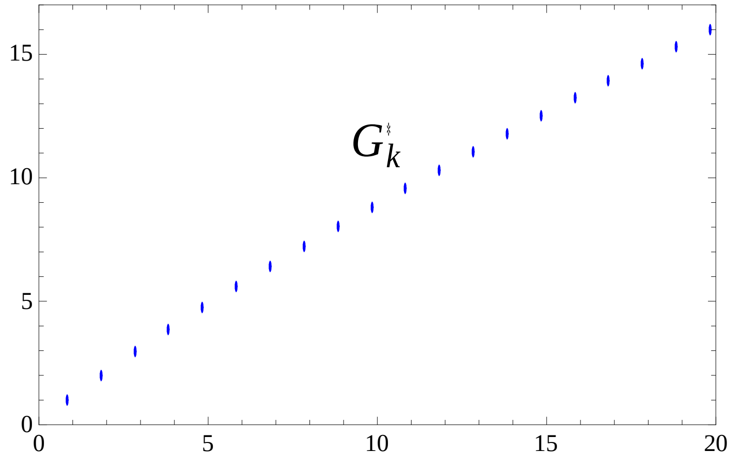


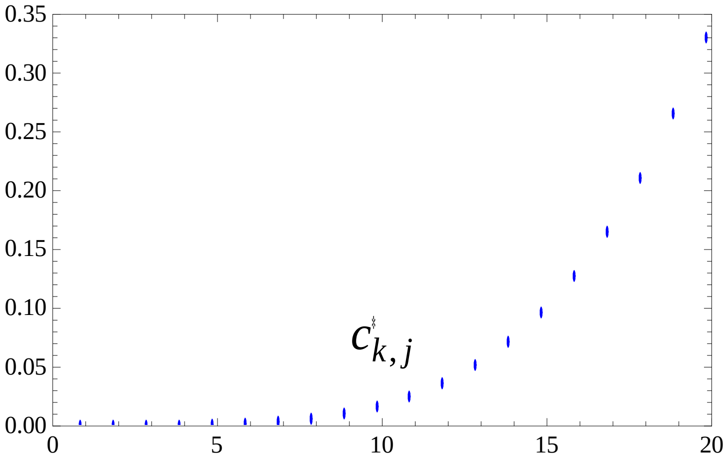




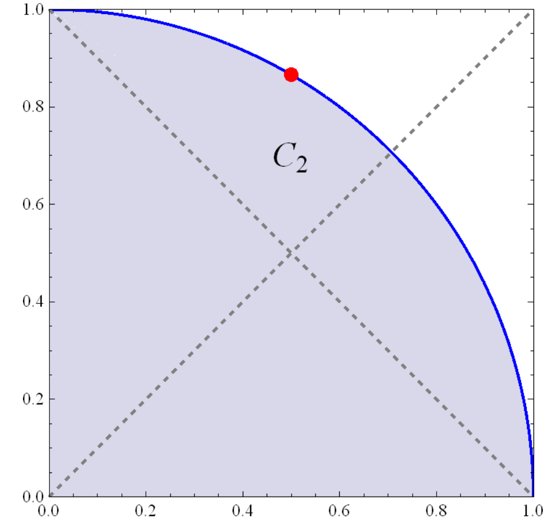
Alpha = 1.2



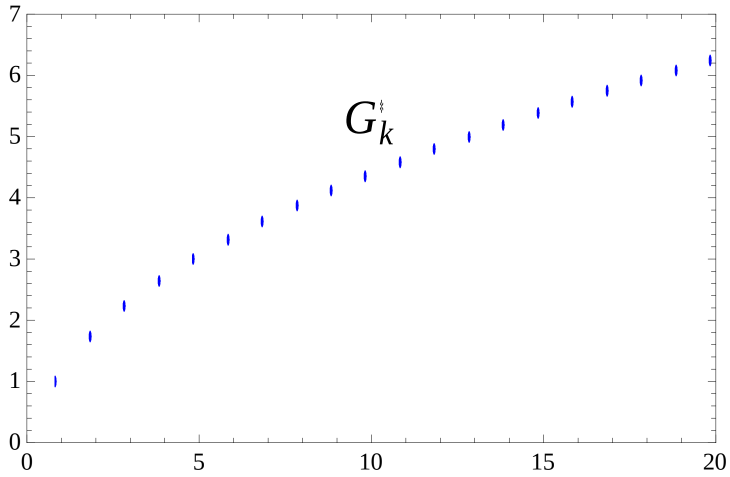




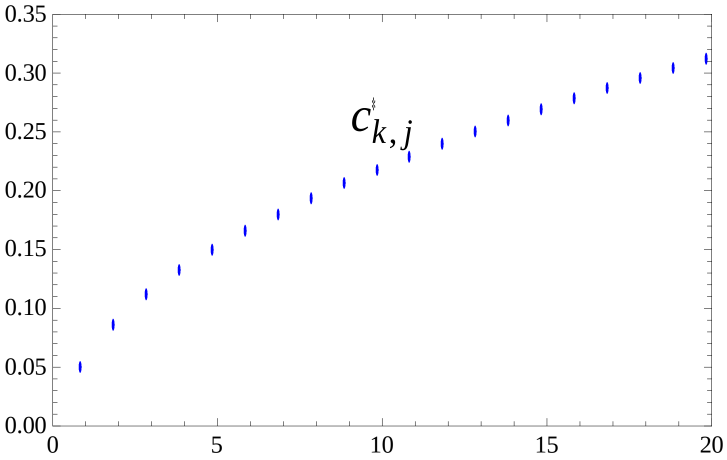
Alpha = 2.0



1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
C_2

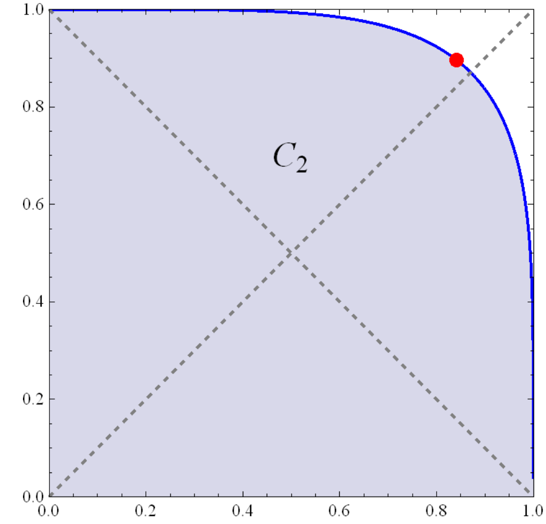


7
6
5
4
3
2
1
0
0
5
10
15
20
$G_k^{\natural}$

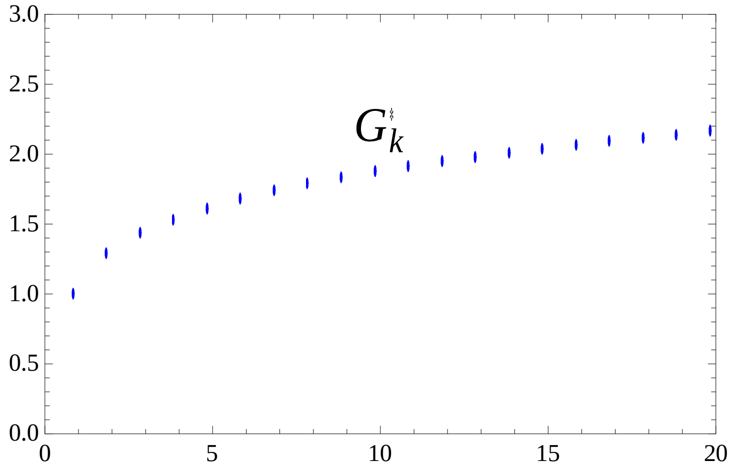


0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0
5
10
15
20
$c_{k,j}^{\natural}$

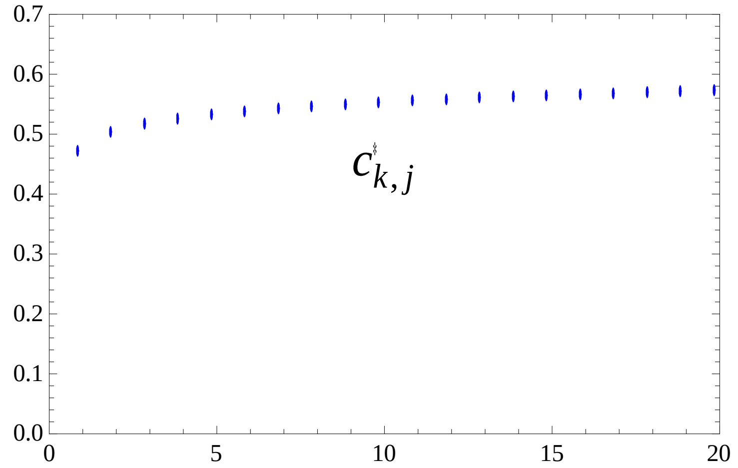
Alpha = 5.0



C_2
0.0
0.2
0.4
0.6
0.8
1.0

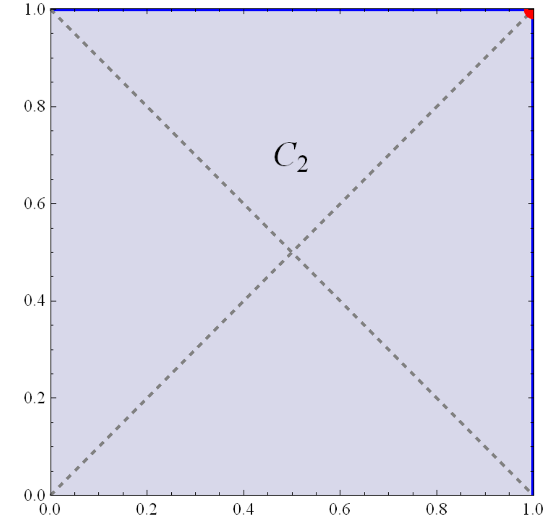


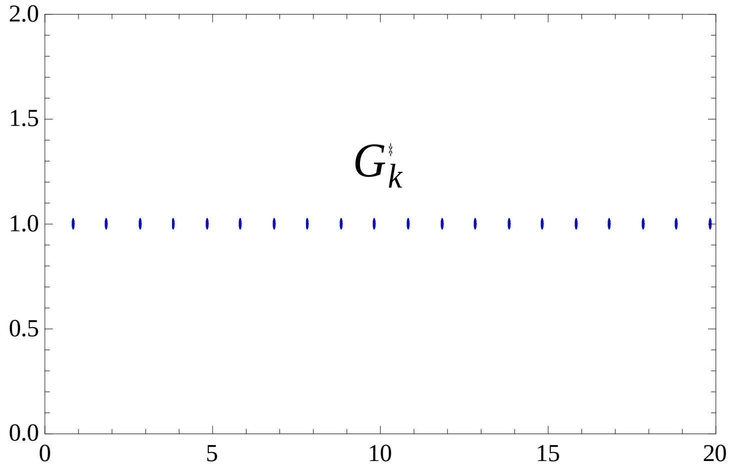
$G^{\ddagger}_k$
0.0
0.5
1.0
1.5
2.0
2.5
3.0
0
5
10
15
20

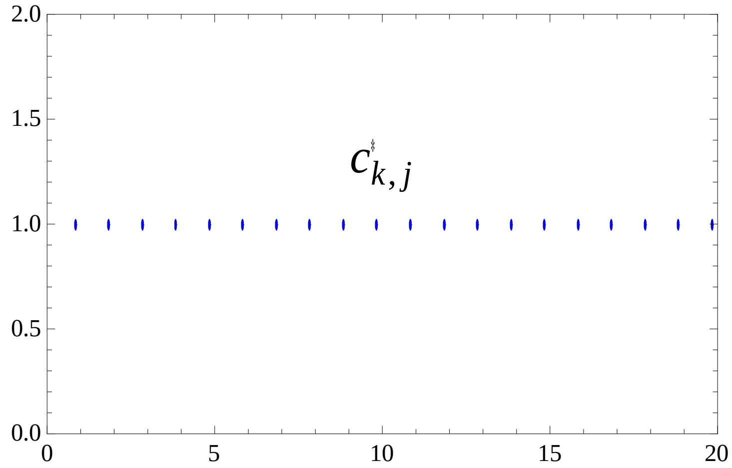


$c^{\ddagger}_{k,j}$
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
5
10
15
20

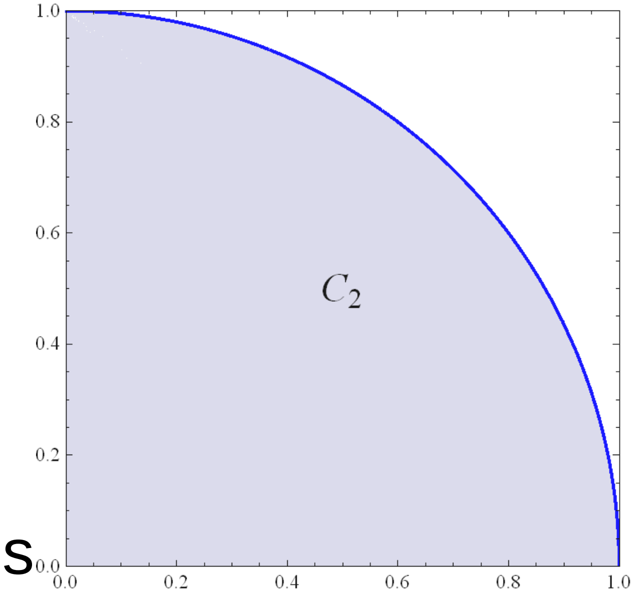
Alpha = infinite (infinite-server queue)







Scalable queue



- Service system where the service capacity scales with number of jobs
- Policy: When there are k jobs with sizes

$$s_1 \geq \ldots \geq s_k$$

choose a rate vector

$$\mathbf{c}_k = (c_{k1}, \ldots, c_{kk}) \in C_k$$

and serve job i with rate c_{ki}

- Assume: Capacity regions C_k compact and symmetric

Optimal scheduling problem (transient system without arrivals)

- Assume that there are n jobs in the system at time 0
- What is the optimal way to make the system empty?
- Objective: Minimize the mean delay (or flow time)
- Define: Flow time for policy ϕ

$$T^{\phi} = \sum_{i=1}^{n} t_i^{\phi}$$

where t_i is the completion time of job i

- Define: Operating policies

$$\Phi_n = \{\phi = (\mathbf{c}_1, \ldots, \mathbf{c}_n) : \mathbf{c}_k \in C_k \text{ for all } k\}$$

Trivial case: One job

- Define:

$$G_1^* = \frac{1}{c_1^*}, \quad c_1^* = \max_{c_1 \in C_1} c_1$$

- Now

$$T^* = \min_{\phi \in \Phi_1} T^{\phi} = s_1 G_1^*, \quad \phi^* = (\mathbf{c}_1^*)$$

General case: n jobs

- Define (recursively):

$$G_k^* = \min_{\mathbf{c}_k \in C_k} g_k(\mathbf{c}_k), \quad g_k(\mathbf{c}_k) = \frac{1}{c_{kk}} \left(k - \sum_{i=1}^{k-1} c_{ki} G_i^* \right)$$

- Theorem [Aalto et al. (2011)]: **If**

$$G_1^* < \ldots < G_n^*$$

then

$$T^* = \min_{\phi \in \Phi_n} T^\phi = \sum_{k=1}^{n} s_k G_k^*, \quad \phi^* = (\mathbf{c}_1^*, \ldots, \mathbf{c}_n^*)$$

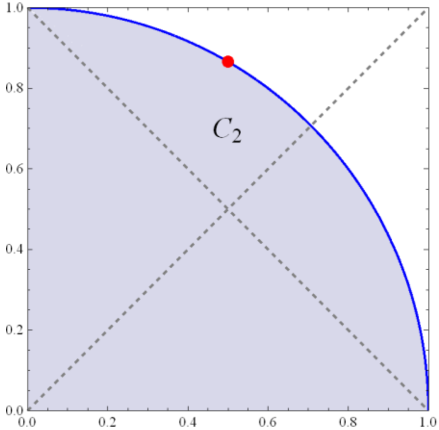
General case: n jobs (cont.)

- In addition,

$$c_{k1}^* \le \ldots \le c_{kk}^* \quad \text{for all } k$$

- The optimal policy applies the SRPT-FM principle
  - the shortest job is served with the highest rate, etc.
- The optimal rate vector does not depend on the absolute sizes of jobs (only on their order)

Alpha-balls



- Let $\alpha \geq 1$ and consider capacity regions

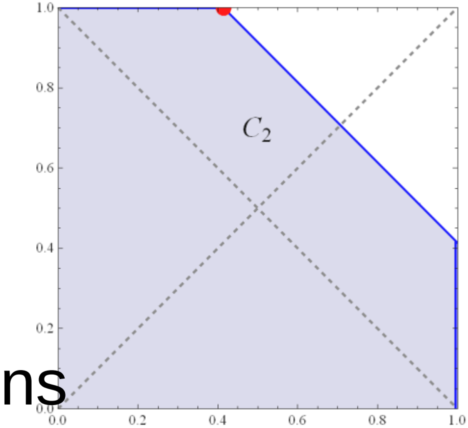
$$C_k = \{\mathbf{c}_k \geq 0 : \sum_{j=1}^{k} c_{kj}^{\alpha} \leq 1\}$$

- Now

$$G_k^* = \left(k^{\frac{\alpha}{\alpha-1}} - (k-1)^{\frac{\alpha}{\alpha-1}} \right)^{\frac{\alpha-1}{\alpha}} \quad \text{(increasing in } k\text{)}$$

$$c_{kj}^* = \left(\frac{G_j^*}{k} \right)^{\frac{1}{\alpha-1}} \quad \text{(increasing in } j\text{)}$$

Symmetric polymatroids



- Let $\gamma_1 < \ldots < \gamma_n$ and consider capacity regions

$$C_k = \{\mathbf{c}_k \geq 0 : \sum_{i \in I} c_{ki} \leq \gamma_{|I|}, I \subset \{1,\ldots,n\}\}$$

- Theorem: If $\gamma_1 > \gamma_2 - \gamma_1 > \ldots > \gamma_n - \gamma_{n-1}$, then

$$G_1^* < \ldots < G_n^* \qquad \text{(increasing in } k\text{)}$$

$$c_{kj}^* = \gamma_{k-j+1} - \gamma_{k-j} \qquad \text{(increasing in } j\text{)}$$

- Optimality result of Sadiq and de Veciana (2010)

Open questions

- Is it possible to make the implicit condition explicit?
- Optimal scheduling problem for a dynamic system with random arrivals?
- Other objective functions?

References

- B. Sadiq and G. de Veciana,
Balancing SRPT prioritization vs opportunistic gain in wireless systems with flow dynamics, in *ITC-22*, 2010

- S. Aalto, A. Penttinen, P. Lassila and P. Osti,
On the optimal trade-off between SRPT and opportunistic scheduling, in *ACM SIGMETRICS*, 2011

- S. Aalto, A. Penttinen, P. Lassila and P. Osti,
Optimal size-based opportunistic scheduler for wireless systems, *Queueing Systems* 72, 2012

The End