Optimizing the Degree Distribution of LT Codes

PAN-NET research seminar 16.3.2006

Outline

- Erasure codes
- LT codes
- The degree distributions
- Optimization of the degree distributions
- Analytic results for $n = 3$ and $n = 4$
- Importance sampling based method for optimizing the degree distribution
- Test results

Erasure codes

- File divided into n blocks is distributed.
- Erasure codes can be used to encode the original data, resulting in $n + m$ blocks
- These blocks are then distributed, any $n' \geq n$ blocks are sufficient to recover the original file (**overhead factor** $f = n'/n \geq 1$)
- With different codes there usually is a trade-off between computational efficiency and f
- E.g. with Reed-Solomon codes, $f = 1$, but computation is inefficient.

The fountain coding principle

1. A server distributes a file consisting of n **blocks** of length l
2. Server encodes blocks into **packets**, and distributes these:
   - 2^n different packets
   - Practically infinite number of these packets can be generated
   - Number of blocks encoded into one packet is the **degree** of packet
3. In ideal case, client needs to collect any n packets in order to decode the original content

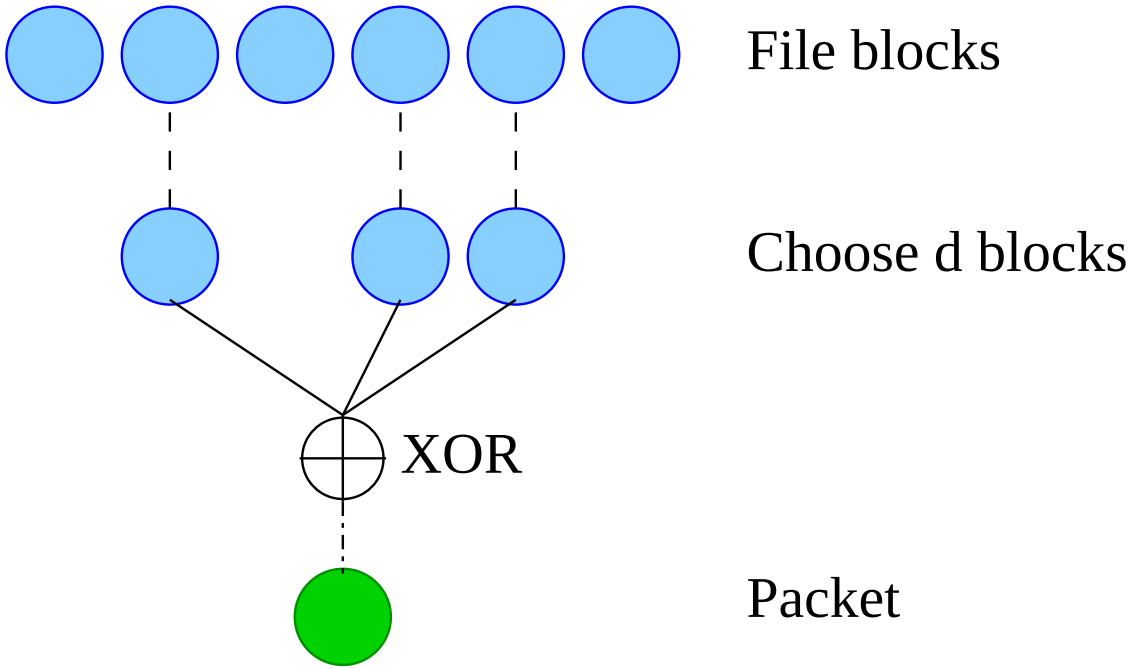
   $\rightarrow$ analogy to filling a bucket under a fountain spraying water drops

- No retransmission of a specific packet is needed
- This is the ideal case, in practice erasure codes *approximate* this.

LT codes

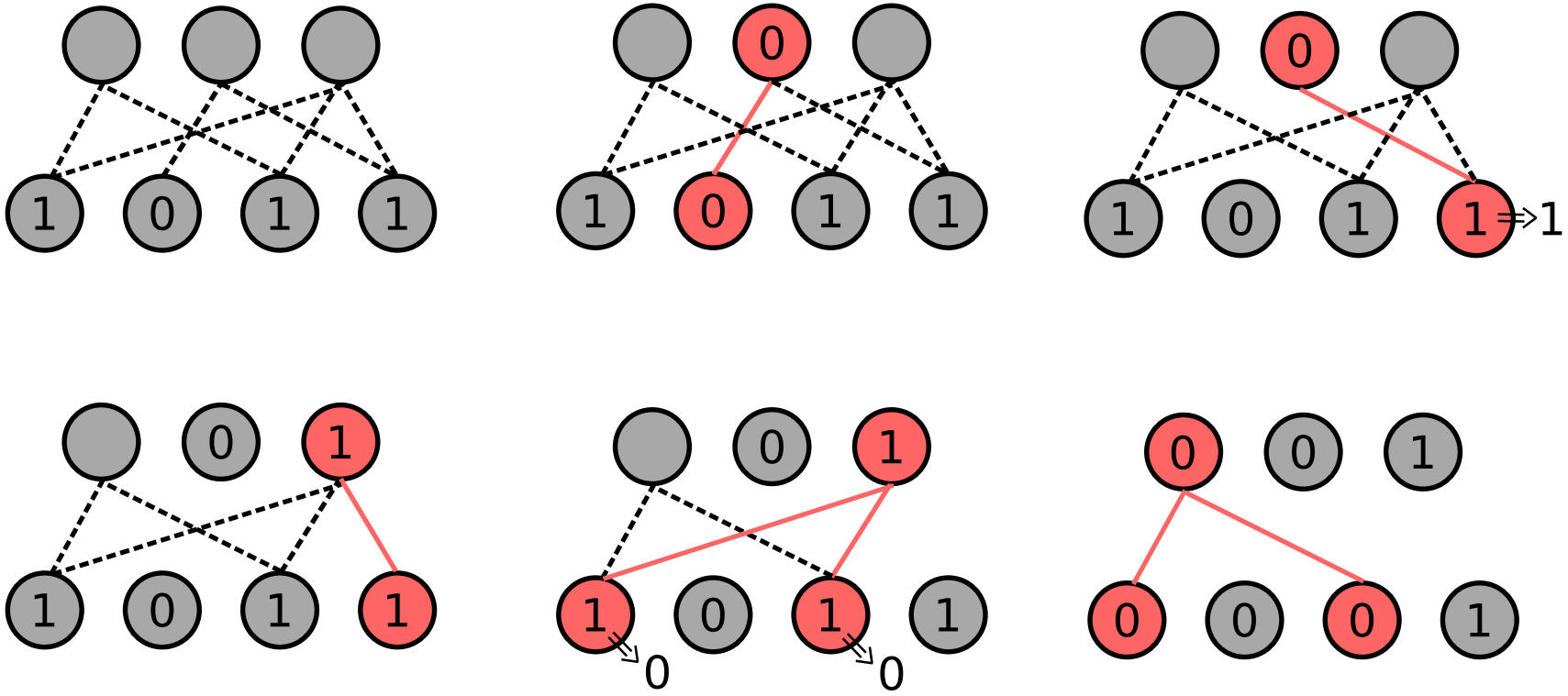
- Efficient codes developed by M. Luby, company Digital Fountain
- Optimal codes for infinite n in terms of overhead
- A **degree distribution** defines the efficiency
- Extension to LT codes with linear time encoding and decoding: **Raptor** codes
- One form of low-density parity-check (LDPC) codes
- Possible applications: realible multicast, P2P, distributed storage...

LT encoding



1. Choose packet degree d from **degree distribution** $\rho(d)$
2. Choose uniformly at random d blocks
3. XOR the selected blocks bitwise

LT decoding

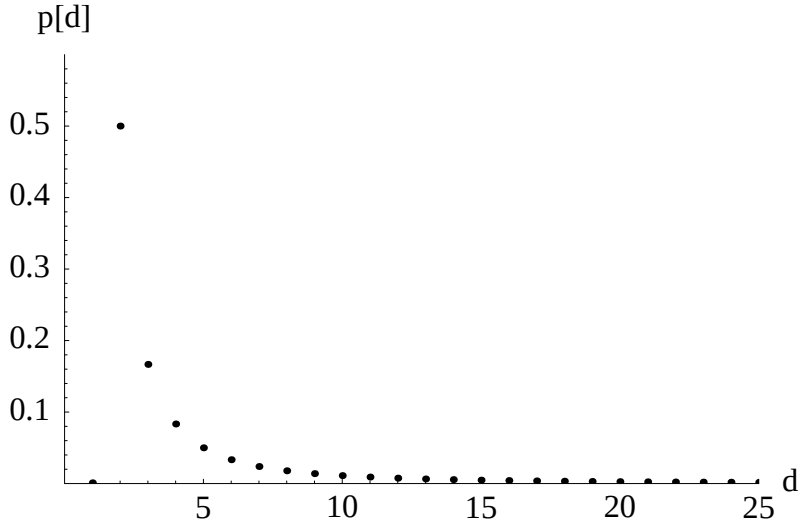


Original file blocks (input symbols) above, encoded packets below.

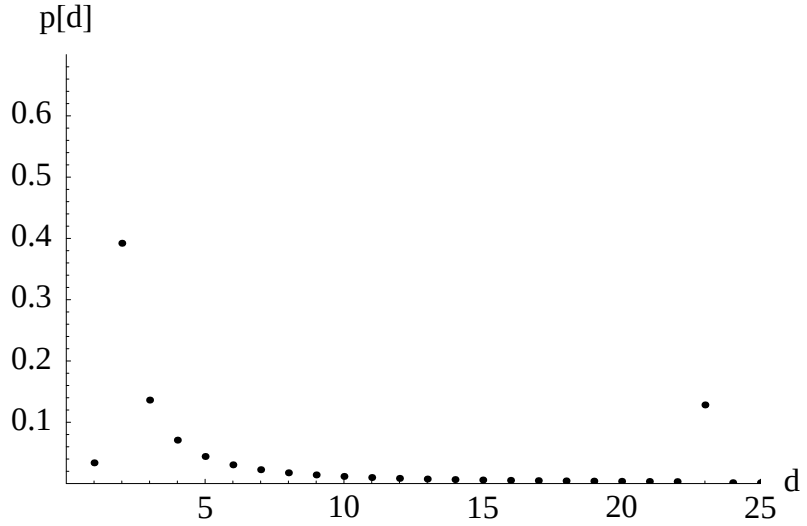
1. Degree one packet starts the decoding.
2. If the packet degree is one, then the neighbor block = packet.
3. XOR the value of the recently recovered block with its neighbors, remove edges.
4. Continue iteratively until done or no more degree one packets.

Degree distributions

- With a good degree distribution the overhead factor is as small as possible
- In expectation, ideal soliton distribution works best.
- From the ideal soliton distribution Luby derived the robust soliton distribution. It is possible to tune this distribution so that for $N \approx 10000$, the overhead is about 5%



Example of ideal soliton distribution



Example of robust soliton distribution, note the spike

Optimal degree distributions in cases $n = 3$ and $n = 4$

- (De)coding process modeled as a Markov's Chain
  - The set of received packets and possibly decoded blocks correnspponds to a state
  - Total number of these states is then 2^{2^n}.
    - 256 for $n = 3$
    - 65536 for $n = 4$
    - $2^{32} \approx 4 \cdot 10^9$ for $n = 5$
  - Next step is to reduce the state set so that state transition matrix $\mathbf{P}$ becomes smaller

State space reduction

- Let's take a file with blocks $\{a, b, c\}$
- Now consider the situation where receiver has reveived a packet with block a and packet of b and c XORred together $\rightarrow$ state $\{a, bc\}$
- With regards to the decoding process, this is the same thing as if we were in state $\{b, ac\}$ or $\{c, ab\}$, so these states can be aggregated into one state
- **Unique representation** for these states is chosen to be $\{a, bc\}$
- Similarly for other states: give a unique representation over permutations of different blocks
- State space can be reduced approximately to square root of 2^{2^n}
  - For $n = 3$ to 15 states, for $n = 4$ to 192 states

Transition matrix generation

- Mathematica is used to automatically perform the state space reduction
- This way automatic generation of transition matrix is possible
- With transition matrix for Markov's chain, average number of steps to absorbing state can be calculated
- This, however, works only for $n = 3$ or 4, for $n = 5$ the number of raw states becomes too large
- The optimized distributions give average overhead of one packet with $n = 3$ and 1.5 packets with $n = 4$.

Optimization for any n

- **Idea**: to estimate the average number of packets using **importance sampling**
- Review of IS:

$$E[h(\mathbf{X})] = \int h(\mathbf{x})p(\mathbf{x})\ d\mathbf{x}. = \int h(\mathbf{x})\frac{p(\mathbf{x})}{g(\mathbf{x})}g(\mathbf{x})\ d\mathbf{x},$$

where $p(\mathbf{x})$ and $g(\mathbf{x})$ are probability distributions. An estimate for expectation $\hat{h}$ can be calculated by drawing samples $\tilde{\mathbf{X}}^{(i)}$ from $g(\mathbf{x})$:

$$\hat{h} = \frac{1}{K}\sum_i h(\mathbf{X}^{(i)}) = \frac{1}{K}\sum_i w(\tilde{\mathbf{X}}^{(i)})h(\tilde{\mathbf{X}}^{(i)}),$$

where we use the *importance ratios*

$$w(\mathbf{x}) = \frac{p(\mathbf{x})}{g(\mathbf{x})}$$

Importance sampling based optimization for LT codes

- Importance sampling allows the calculation of an expectation using a different distribution than the original one
- We propose an optimization method for LT codes using this fact, the estimate for average number of packets needed for decoding is:

$$\widehat{R}\left(\mathbf{q}\right) = \frac{1}{m}\sum_{k=1}^{m} R_k \prod_i \left(\frac{q_i}{p_i}\right)^{n_i^{(k)}},$$

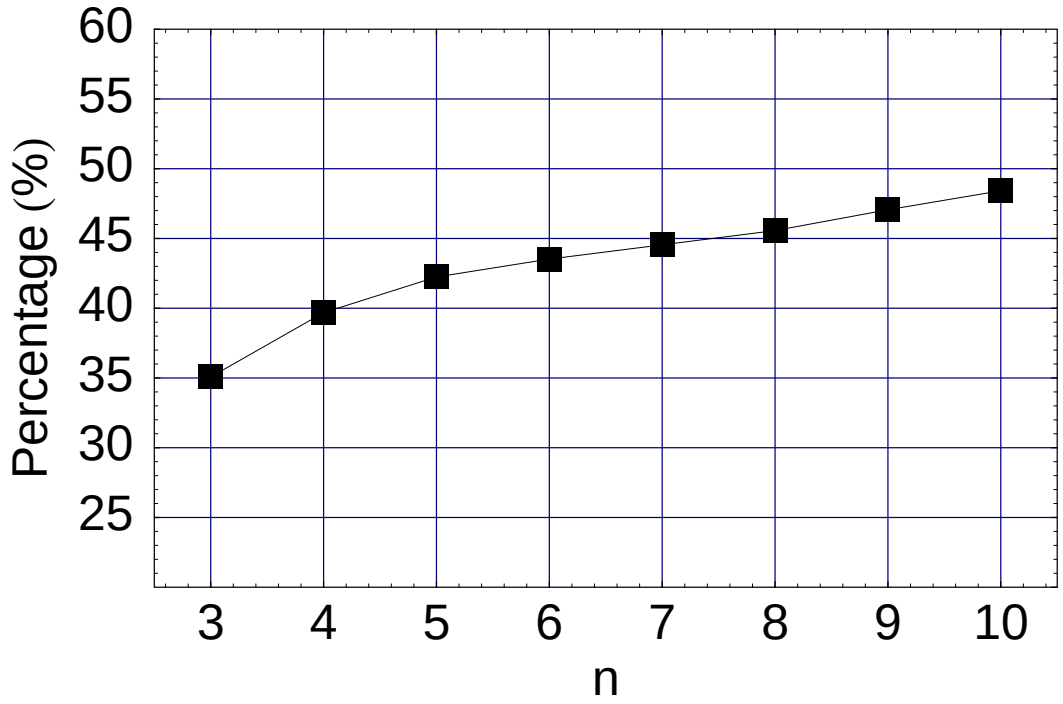
  where $n_i^{(k)}$ denotes the number of packets of degree i, $\mathbf{q}$ and $\mathbf{p}$ are degree distributions, R_k is the number of packets needed for decoding
- This means that we can generate samples of LT process with one probability distribution and use the presented equation to calculate the expectation with a different degree distribution

Importance sampling based optimization for LT codes

- We implemented an iterative algorithm, which optimizes the estimate by the method of steepest descent (i.e. gradient method)
- The accuracy of the estimate can be controlled trough the gradient, i.e., samples are generated until the variance of the gradient is small enough
- The algorithm takes some degree distribution as input and outputs a better one, if possible
- Present implementation has been tested for $n \leq 100$, in order to keep the simulation and optimization times reasonable

Results

- We optimized point distributions (i.e. parameters are the probabilities for each degree) for $n = 3, \ldots 10$.
- Plot of average overhead percentages from 100000 simulations with the optimized distributions:

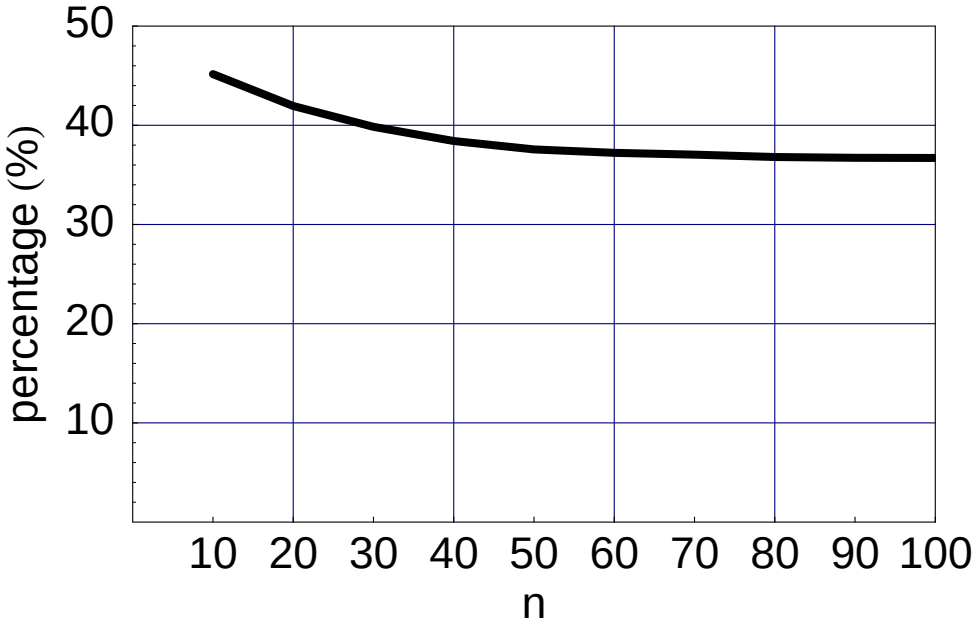


Results

- We tested geometric forms with one and three parameters, optimizing the parameters

$$p_i = e^{-\eta i} \text{ and } p_i = e^{-\eta_1 i} + \eta_2 e^{-\eta_3 i},$$

- Overhead percentages again from 100000 simulations with optimized distributions (roughly the same for both forms!)

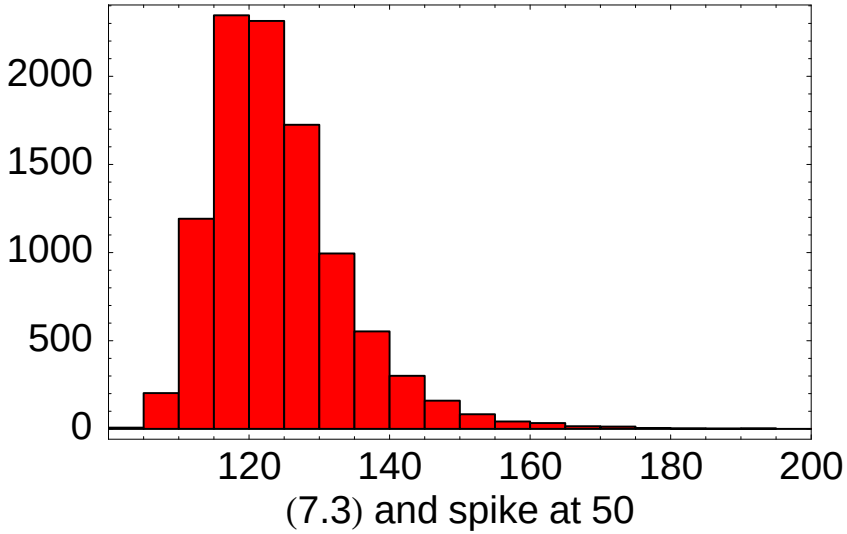


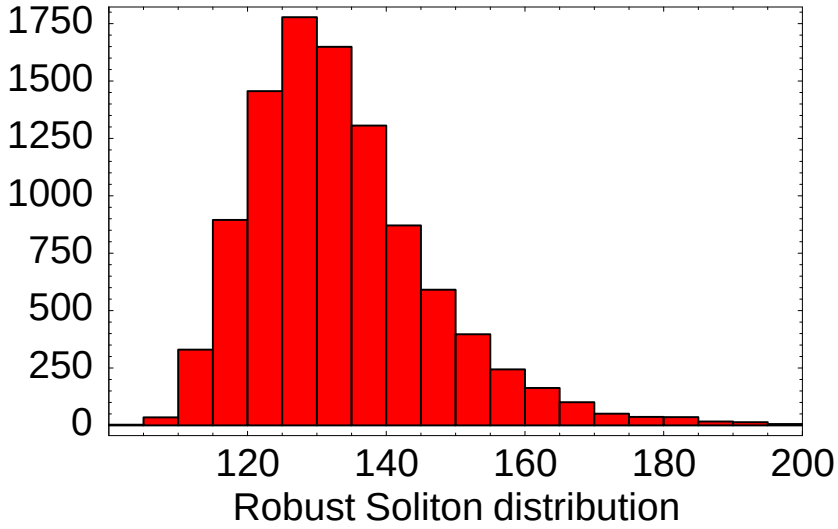
Results with $n = 100$

- We tried to create a more efficient form from soliton distributions
- Optimized parameters are the probabilities for degrees one and two, and additionally for degree 50. Otherwise the distribution is soliton distribution

Distribution	Avg	Std
Degrees one and two optimized	125	13
Degrees one and two + spike at 50	124	10
Soliton	170	70
Robust soliton	130	13

Histograms of simulations with $n = 100$





Numbers of packets needed with the best form we found and with robust soliton distribution.

Tests with $n = 1000$

- We tested the optimized distributions for $n = 100$ with simulations of the LT process when $n = 1000$

Distribution	Avg	Std
Degrees one and two optimized	1130	84
Degrees one and two + spike at 100	1121	37
Robust soliton	1124	57

- To our surprise, the optimized results for $n = 100$ perform well.

Conclusions

- Our proposed optimization strategy works
- We were able to generate better results than with the robust soliton distribution for $n = 100$ and even for $n = 1000$
- Open questions:
  - What is the best form of degree distribution?
  - Is the spike really needed?