Balanced fairness

Adapted from a poster presentation by T. Bonald and A. Proutière

(in Conference on Stochastic Networks, Stanford Business School, 2002)

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Outline

- Introduction, bandwidth sharing, insensitivity . . .
- Network models: physical network, Whittle queueing network
- Balance property, balance function, balanced fairness
- Examples: lines, grids, trees, cycles
- How to calculate the balance function: parking lot
- Normalization constant and performance measures
- Recursive algorithm for the normalization constant and throughputs
- Examples: line, parking lot, trees; comparison with bounds
- Limited access rate
- State dependent arrival rates, adaptive routing
- Open problems, future research

Introduction

- Traffic theory and network design
  - The objective is to characterize the *demand–capacity–performance* relationship accounting random flow arrivals.
  - Can we operate the network so that this relationship depends only on simple traffic parameters – like the Erlang formula in telephone networks?

Bandwidth sharing and the notion of utility

- Static bandwidth sharing
  - Max-min fairness (Bertsekas & Gallager, 1987)
  - Proportional fairness (Kelly, 1998)
  - More general utility-based allocations (Mo & Walrand, 2000)

Bandwidth sharing (continued . . .)

- Dynamic bandwidth sharing with random traffic
  - A single link: explicit results for fair sharing (Ben Fredj & al., 2001)
    - ∗ insensitive to any traffic characteristics
  - A homogeneous line, grid or hypercube (Bonald & Massoulié, 2001)
    - ∗ explicit results for proportional fairness only
    - ∗ also insensitive to any traffic characteristics

Questions posed by Bonald and Proutière

Can we characterize the set of allocations leading to insensitivity?

Can we approximate the performance of usual allocations (max-min, TCP,. . .)?

Insensitivity in stochastic networks

- Insensitivity: the state distribution does not depend on the service time distributions.
- Literature: Barbour, Kelly, Baccelli, Baskett et al., Schassberger, Whittle, Miyazawa, Henderson, Burman, . . .

Insensitivity (continued . . .)

- Insensitivity in GSMP (Schassberger, 1977)
  - Partial balance is equivalent to insensitivity.
  - But this does not cover simple stochastic networks – not even the M/GI/PS queue.
  - Usual networks are covered by more general processes, R-GSMP (Schassberger, 1986, Miyazawa, 1993),
    - ∗ partial balance is equivalent to product form decomposability, which implies insensitivity,
    - ∗ the converse is not true in general.

Insensitivity (continued . . .)

- Bonald and Proutière
  - proved that for state dependent processor sharing networks partial balance is equivalent to insensitivity,
  - characterized insensitive allocations in data networks.
- Balanced fairness
  - Flow level performance metrics are independent of detailed traffic characteristics: flow size distribution, distribution of the number of flows per session, correlation between successive flow sizes and think-time durations . . . as far as session arrivals are Poisson.

Physical network model

- Network: L links
  - capacities $C_1, \ldots, C_L$
- Flows: K flow types (classes)
  - route r_i (a set of links)
  - arrival rate λ_i
  - mean flow size $1/\mu_i$
  - network state $x = (x_1, \ldots, x_K)$: number of flows in each class

Whittle queueing network model (session/flow level)

- Network: K processor sharing nodes
  - Each node represents a flow class (route) or a think time.
  - Node is served with capacity $\phi_i(x)$ (depends on the network state).
  - External *Poisson* arrival rate to each node ν_i
    - represents *session* arrivals starting with flow i.
  - After completion of flow i the session moves to another node j
    - probability p_{ij}: session generates another flow or goes idle

Constraints

Total arrival rate of type i flows determined by

$$\lambda_i = \nu_i + \sum_j p_{ij}\lambda_j, \qquad \rho_i = \lambda_i/\mu_i$$

Bandwidth constraint

$$\sum_{i\in\mathcal{F}_l} \phi_i(x) \le C_l, \quad \forall l,$$

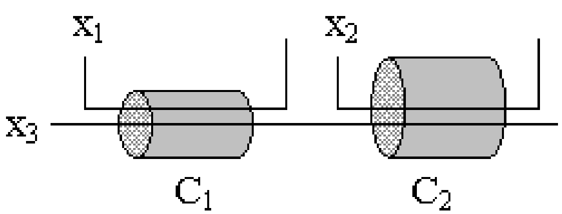
Stability condition

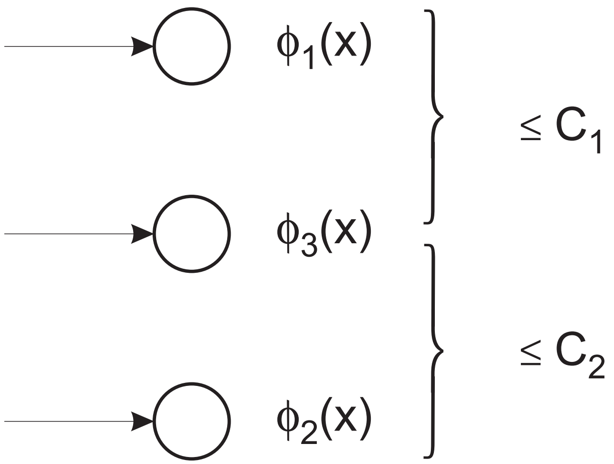
$$\sum_{i\in\mathcal{F}_l} \rho_i < C_l, \qquad \forall l,$$

where $\mathcal{F}_l =$ the set of flow classes going through link l.

Example 1

- A data network represented as a processor sharing network

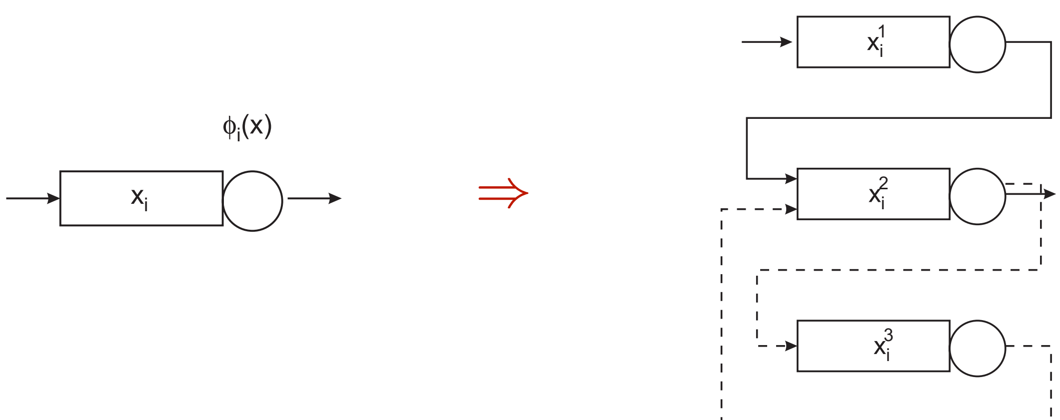




Example 2

- From Poisson arrivals, i.i.d. exponential services and Bernoulli routing to very general traffic characteristics: one can represent any type of *session* (succession of correlated flows and think times) by adding new nodes.

- For example, a session on a route with 2 flows types with Erlang-2 and exponential sizes. The exponential flow is repeated separated by an exponential think time:



$$\phi_i^1(x) = \frac{x_i^1}{x_i^1 + x_i^2}\,\phi_i(x)$$

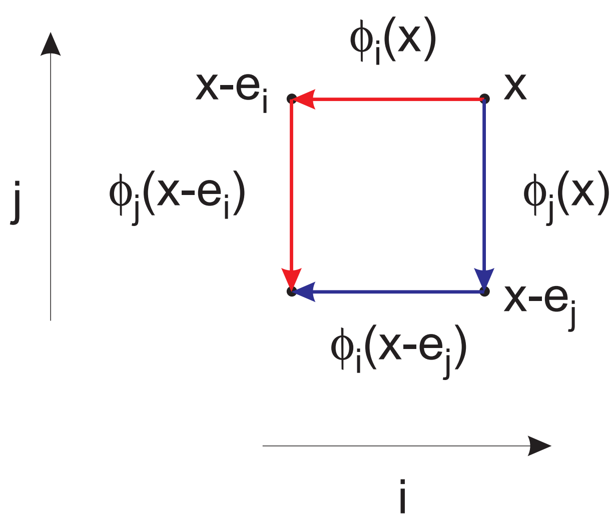
$$\phi_i^2(x) = \frac{x_i^2}{x_i^1 + x_i^2}\,\phi_i(x)$$

$$\phi_i^3(x) = x_i^3$$

Insensitive PS networks: balance property

- Whittle network: PS network satisfying the *balance property*

$$\frac{\phi_i(x-e_j)}{\phi_i(x)} = \frac{\phi_j(x-e_i)}{\phi_j(x)}, \qquad \forall\, i,j,\; x_i>0,\; x_j>0,$$



$$\phi_i(x)\phi_j(x-e_i) \;=\; \phi_j(x)\phi_i(x-e_j)$$

"left path" = "right path"

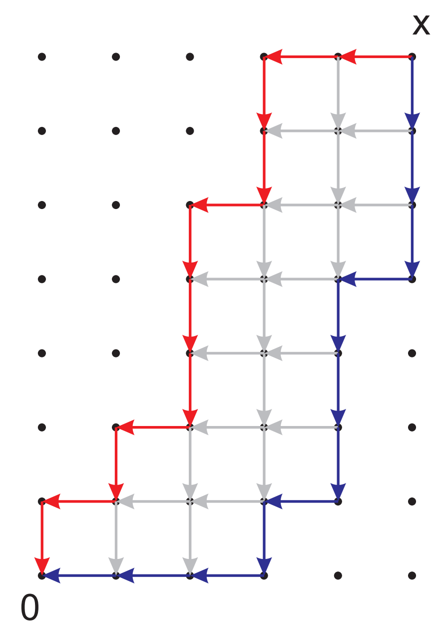
Balance property (continued . . .)

- Remark: if for all classes i the allocation ϕ_i depends only on x_i and not on the number of flows in other classes, the system is balanced;
  - because then $\phi_i(x - e_j) = \phi_i(x)$
  - e.g., multibitrate systems

Balance function

- Let p be a path of length n from state 0 to state x with n flows

$$p = (x, x - e_{k_1}, x - e_{k_1} - e_{k_2}, \ldots, x - e_{k_1} - \ldots - e_{k_{n-1}}, 0)$$



- When the balance property holds the product

$$\phi_{k_1}(x)\phi_{k_2}(x - e_{k_1}) \cdots \phi_{k_n}(x - e_{k_1} - \ldots - e_{k_{n-1}})$$

is independent of the path between 0 and x.

- The inverse of the product is called the *balance function*

$$\Phi(x) = \frac{1}{\phi_{k_1}(x)\phi_{k_2}(x - e_{k_1}) \cdots \phi_{k_n}(x - e_{k_1} - \ldots - e_{k_{n-1}})}$$

Balance function (continued . . .)

- Any positive function $\Phi(x)$ will do as a balance function and defines a balanced allocation.

- Given the balance function, the capacity allocated to class i is

$$\phi_i(x) = \frac{\Phi(x - e_i)}{\Phi(x)}, \qquad \forall\, i,\ x_i > 0.$$

- An acceptable allocation has to satisfy the capacity constraints

$$\sum_{i \in \mathcal{F}_l} \phi_i(x) \le C_l \ \Rightarrow \sum_{i \in \mathcal{F}_l} \frac{\Phi(x-e_i)}{\Phi(x)} \le C_l \ \Rightarrow \ \Phi(x) \ge \frac{1}{C_l} \sum_{i \in \mathcal{F}_l} \Phi(x - e_i),\ \forall l$$

- Even then there are an infinity of different balanced allocations.

Balanced fairness

- *Balanced fairness* refers to the balanced allocation with the property that in each state *at least one link is saturated*.

- Balanced fair allocation is *uniquely* defined by the recursion

$$\Phi(x) = \max_l \left\{ \frac{1}{C_l} \sum_{k \in \mathcal{F}_l} \Phi(x - e_k) \right\}$$

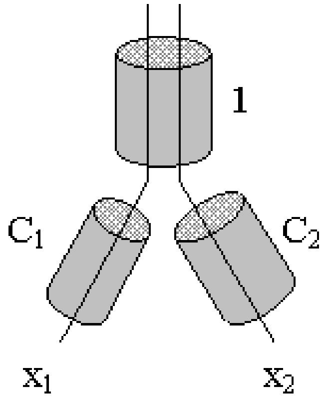
  that is, $\Phi(x)$ is reduced until a capacity constraint is encountered.

- Balanced fairness defines the most efficient balanced allocation (not necessarily Pareto efficient).

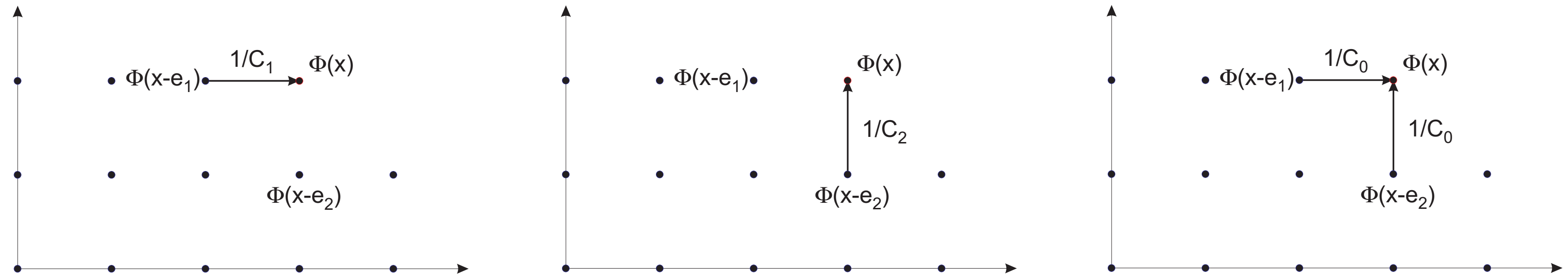
Balanced fairness: example of the basic recursion

- The recursion

$$\Phi(x) = \max_{l} \left\{ \frac{1}{C_l} \sum_{k:\, l \in r_k} \Phi(x - e_k) \right\}$$



- In the case of a 2-branch tree the maximum of these:

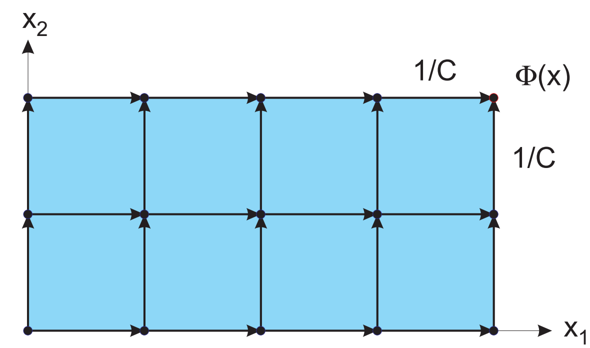


Balanced fairness: a trivial example

- A single link of capacity C shared by two classes.

- The balance function

$$\Phi(x) = \text{(number of paths)} \times \frac{1}{C^{\text{(path length)}}} = \binom{x_1+x_2}{x_1} \frac{1}{C^{x_1+x_2}}$$

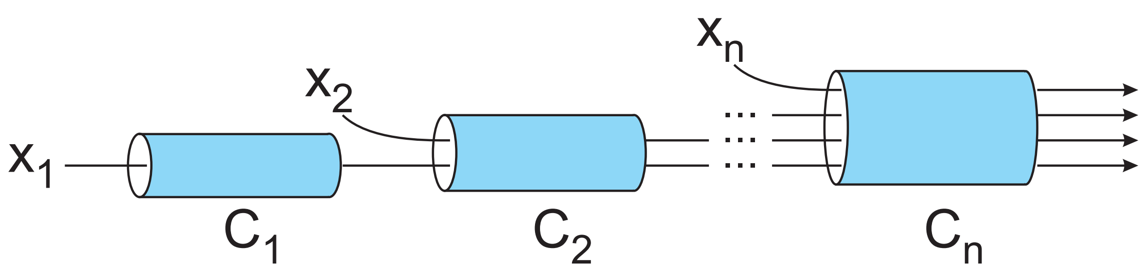


- Capacity allocation: $\phi_i(x) = \frac{\Phi(x-e_i)}{\Phi(x)} = \frac{x_i}{x_1+x_2}C, \ i = 1, 2.$

- Each flow is allocated the capacity $\frac{C}{x_1+x_2}$: a single PS system.

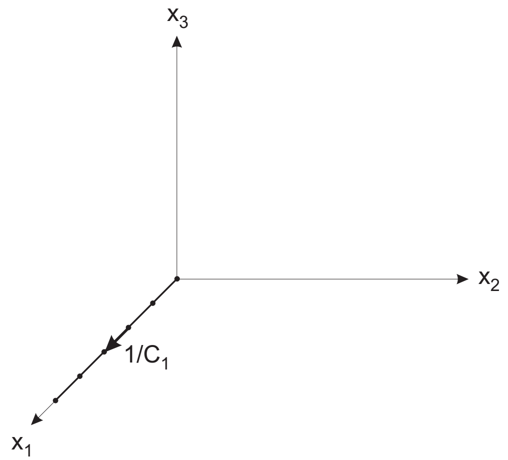
How to determine the balance function: parking lot

- Parking lot topology

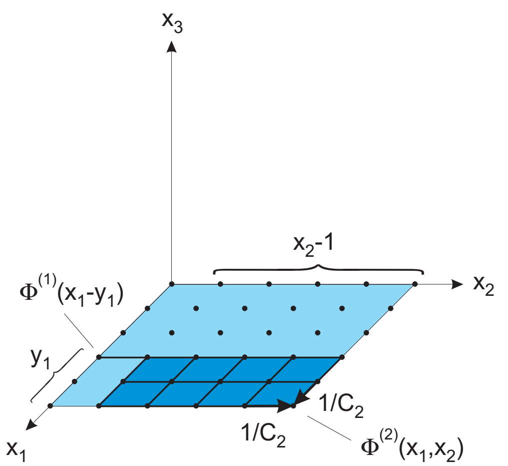


- Special case of a multi-level tree

How to determine the balance function (continued . . .)



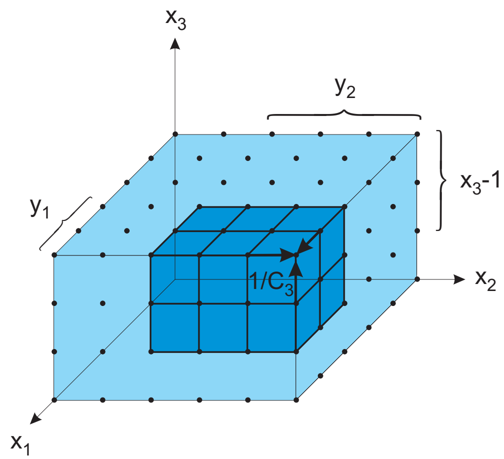
$$\Phi^{(1)}(x_1) = \frac{1}{C_1^{x_1}}$$



$$\Phi^{(2)}(x_1, x_2) =$$

$$\sum_{y_1 \le x_1} \binom{y_1 + x_2 - 1}{y_1} \frac{\Phi^{(1)}(x_1 - y_1)}{C_2^{y_1 + x_2}}$$

How to determine the balance function (continued . . .)



$$\Phi^{(n)}(x_1, \ldots, x_n) =$$

$$\sum_{y_1 \le x_1, \ldots, y_{n-1} \le x_{n-1}} \binom{y_1 + \cdots + y_{n-1} + x_n - 1}{y_1, \ldots, y_{n-1}} \frac{\Phi^{(n-1)}(x_1 - y_1, \ldots, x_{n-1} - y_{n-1})}{C_n^{y_1 + \cdots + y_{n-1} + x_n}}$$

State distribution under balanced fairness

- In a dynamic setting where flows arrive and are transferred across the network until completed the network state x is a random variable.

- The invariant measure (unnormalized distribution) is

$$\pi(x_1, \ldots, x_K) = \Phi(x_1, \ldots, x_K)\rho_1^{x_1} \cdots \rho_K^{x_K}$$

  - $\rho_i = \lambda_i/\mu_i$ is the load of flow class i ($1/\mu_i$ is the mean flow size),
  - ρ_i has the dimension of rate, e.g. kbit/s.

State distribution (continued . . .)

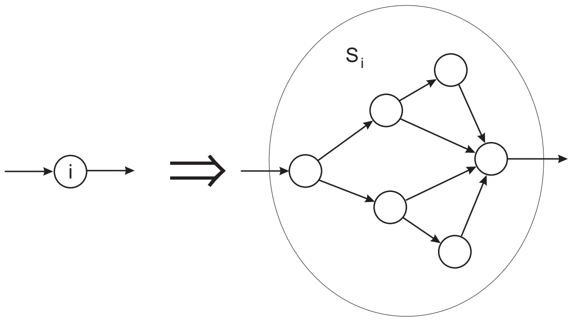
- If $p_{ij} = 0$ for $i \neq j$, balance property implies detailed balance and the form of the invariant measure is obvious but it holds also for the queueing network.

- The invariant measure is insensitive to any traffic characteristics.
  - Whittle networks are the only insensitive PS networks.

Insensitivity

- From exponential service times (flow sizes) to general distribution.

- Replace node i by a set of sub-nodes S_i such that each node $\iota \in S_i$ represents an exponential phase of a phase type service time distribution;



  - let y_ι be the number of customers in sub-node ι, $\sum_{\iota \in S_i} y_\iota = x_i$,
  - total capacity $\phi_i(x)$ allocated to node i is equally shared by all customers in the node no matter which phase (sub-node) they are in,
  - the capacity $\psi_\iota(y)$ allocated to sub-node ι is then $\psi_\iota(y) = \dfrac{y_\iota}{x_i}\phi_i(x)$.

Insensitivity (continued . . .)

- This allocation is still balanced.

- If the $\phi_i(x)$ are balanced by $\Phi(x)$ then the $\psi_\iota(y)$ are balanced by $\Psi(y)$,

$$\Psi(y) = \prod_{i=1}^{K} \binom{x_i}{y_\iota, \iota \in S_i} \Phi(x)$$

- It is readily verified that $\psi_\iota(y) = \Psi(y - e_\iota)/\Psi(y)$.

- The system is another Whittle network with the invariant measure

$$\chi(y) = \Psi(y) \prod_{i=1}^{K} \prod_{\iota \in S_i} \rho_\iota^{y_\iota}$$

Insensitivity (continued . . .)

- The aggregate invariant measure of the node states is obtained by summing over the states of the sub-nodes with the constraints $\sum_{\iota \in S_i} y_\iota = x_i$,

$$\pi(x) = \Phi(x) \prod_{i=1}^{K} \sum_{\substack{\sum y_\iota = x_i \\ \iota \in S_i}} \binom{x_i}{y_\iota, \iota \in S_i} \prod_{\iota \in S_i} \rho_\iota^{y_\iota} = \Phi(x) \prod_{i=1}^{K} (\sum_{\iota \in S_i} \rho_\iota)^{x_i} = \Phi(x) \prod_{i=1}^{K} \rho_i^{x_i}$$

- This is the same as in the original network.

- The invariant measure is insensitive to the replacement of exponential service times by phase type distributions $\Rightarrow$ *general insensitivity*.

- Remark: the above is valid even if node-i service time is not exponential: the aggregate invariant measure is insensitive to the division of any node i into sub-nodes equally sharing the capacity $\phi_i(x)$.

Throughput

- The most important performance measure for elastic traffic is the flow throughput, defined as (mean flow size)/(mean flow response time)

$$\gamma_i = \frac{1/\mu_i}{\mathrm{E}[T_i]} = \frac{\rho_i}{\mathrm{E}[x_i]}$$

  – the latter form follows from expanding the expression by λ_i and then applying Little's theorem.

- A direct calculation using $\chi(y)$ shows that if node i is divided into a set S_i of sub-nodes equally sharing the capacity $\phi_i(x)$, all the sub-nodes have the same throughput

$$\frac{\rho_\iota}{\mathrm{E}[x_\iota]} = \frac{\rho_{\iota'}}{\mathrm{E}[x_{\iota'}]} \quad \left(= \frac{\sum_{\iota \in S_i} \rho_\iota}{\mathrm{E}[\sum_{\iota \in S_i} x_\iota]} = \frac{\rho_i}{\mathrm{E}[x_i]} = \gamma_i\right) \qquad \forall \iota, \iota' \in S_i$$

Linearity of the conditional response time

- Denote by σ_i the amount of service required at node i. Then,

$$\gamma_i = \frac{s}{\mathrm{E}\,[T_i \,|\, \sigma_i = s]}$$

Proof: Replace node i by a set of nodes equally sharing the capacity $\phi_i(x)$, each node corresponding to a different value of σ_i. All the sub-nodes, including the one with $\sigma_i = s$, have a throughput equal to γ_i.

- Conversely,

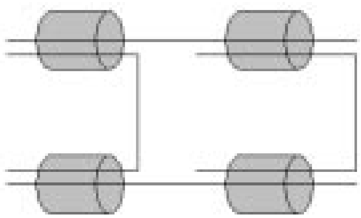
$$\mathrm{E}\,[T_i \,|\, \sigma_i = s] = \frac{s}{\gamma_i}$$

Comparison with utility-based allocations

- Allocations considered so far are based on the notion of utility
  - the allocation is defined by the solution of an optimization problem

$$\begin{cases} \max & \sum_i x_i U(\phi_i/x_i) \\ & \sum_{i \in \mathcal{F}_l} \phi_i \leq C_l \end{cases}$$

- Balanced fairness coincides with proportional fairness on homogeneous hypercubes.



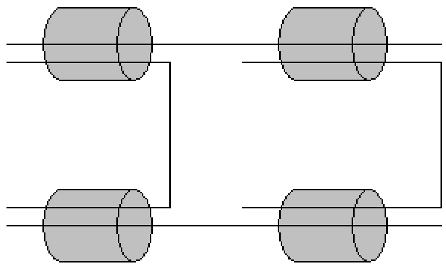
$$\phi_1(x) = \phi_2(x) = \frac{x_1 + x_2}{x_1 + x_2 + x_3 + x_4}, \qquad \phi_3(x) = \phi_4(x) = \frac{x_3 + x_4}{x_1 + x_2 + x_3 + x_4}.$$

Application to specific network topologies

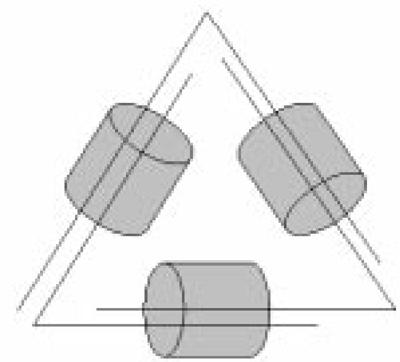
- Utility-based allocations are always sensitive except for proportional fairness in homogeneous hypercubes.
  - For max-min fairness, performance is sensitive.
- Balanced fairness is compared with max-min fairness on different network topologies: lines, grids, hypercubes, trees, hypercycles.
  - User performance is determined by the mean flow throughput = mean flow size / mean flow duration.
- *Balanced fairness provides a good approximation of max-min fairness!*

Definition of some topologies

- A *hypercube* of dimension K is a network made of K classes of routes (called directions) such that the set of links is the set of intersections of K routes of different classes.



- A *hypercycle* is a network made of N routes such that the set of links is the set of intersections of $N-1$ routes.



Homogeneous hypercubes

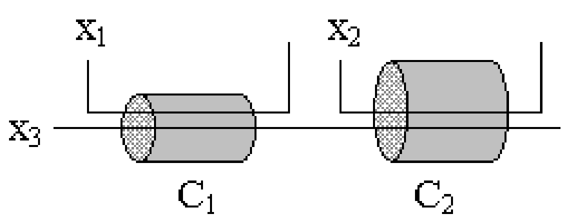
- K directions; sets of routes in each direction $\mathcal{D}_1, \ldots, \mathcal{D}_K$.
- Contains as special cases homogeneous lines and grids.
- Balance function and capacity allocation

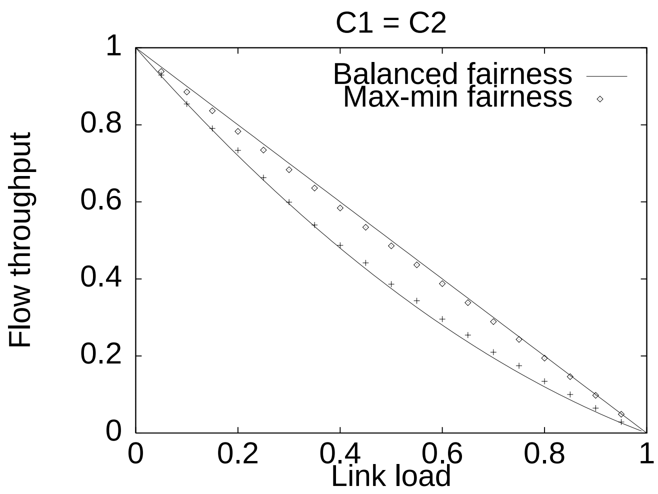
$$\Phi(x) = \binom{\sum_i x_i}{\sum_{i:r_i \in \mathcal{D}_1} x_i, \ldots, \sum_{i:r_i \in \mathcal{D}_K} x_i} \frac{1}{C^{\sum_i x_i}}, \qquad \phi_j(x) = \frac{\Phi(x - e_j)}{\Phi(x)} = \frac{\sum_{i:r_i \in \mathcal{D}_j} x_i}{\sum_i x_i} C$$

Heterogeneous lines

- Heterogeneous lines (minimum capacity as a unit of bandwidth):

$$\Phi(x) = \sum_{y_1+y_2 \le x_3} \binom{x_1+y_1-1}{y_1}\binom{x_2+y_2-1}{y_2}\frac{1}{C_1^{x_1+y_1}C_2^{x_2+y_2}}$$

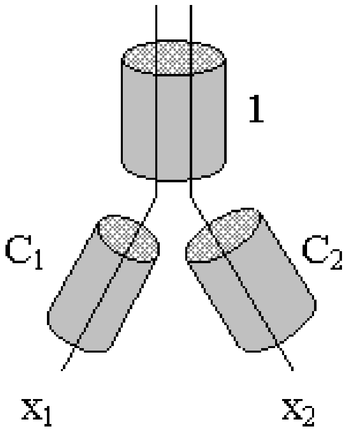


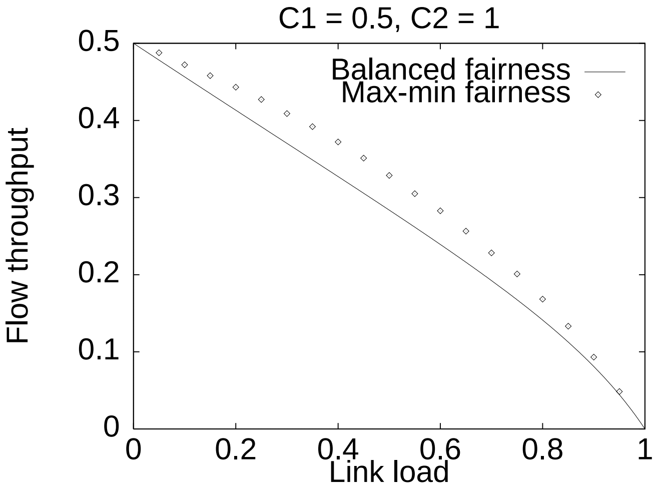


Trees

- The balance function is known for a general tree.
- In particular, consider a two-branch tree:

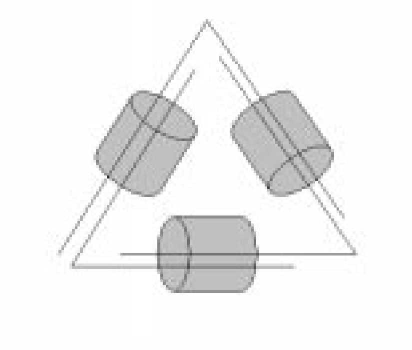
$$\Phi(x) = \sum_{y_1 < x_1} \binom{y_1 + x_2 - 1}{y_1} \frac{1}{C_1^{x_1 - y_1}} + \sum_{y_2 < x_2} \binom{y_2 + x_1 - 1}{y_2} \frac{1}{C_1^{x_2 - y_2}}$$

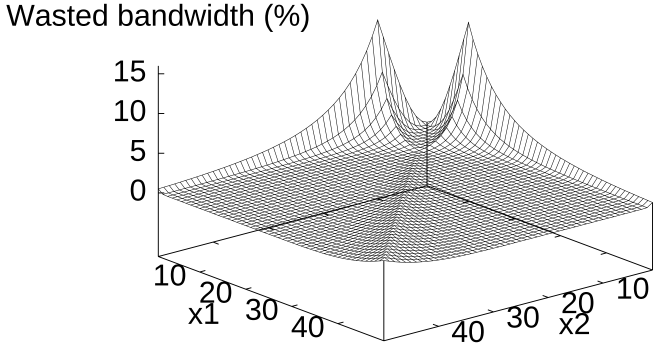


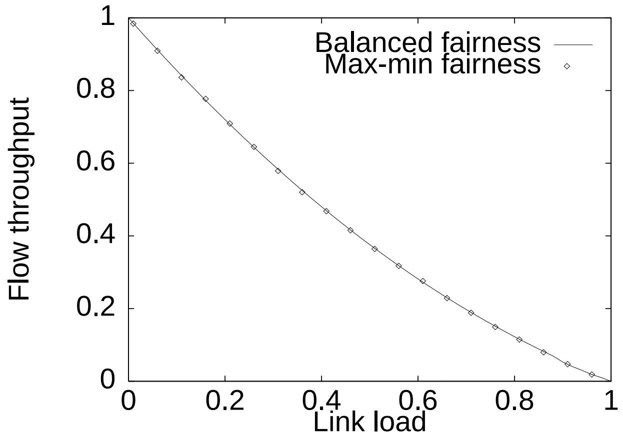


Cycles

- There is no closed expression for the balance function
  - but it can be computed recursively.
- Balanced fairness is not Pareto-efficient on hypercycles
  - but the performance is not deteriorated.







Some bounds for the throughput

- Bonald and Proutière have shown that a store-and-forward network provides a simple lower bound for the throughput:
  - flows are thought to be sent as “packets”; the nodes are PS servers
  - inverse of the sum of the inverse throughputs “$(C_l - \rho_l)^{-1}$”
- Parking lot provides another lower bound
  - all the capacity constraints outside the considered route are relaxed
  - the cross traffic streams are made less constrained than in reality
- Deterministic approximation gives an upper bound
  - all constraints outside the route are made as tight as stability allows
  - cross traffic streams become deterministic: subtract their load from the capacities on the main route; the bottleneck determines the throughput

Normalization constant

- An important role is played by the normalization constant

$$G(\rho) = G(\rho_1, \ldots, \rho_K) = \sum_{x_1=0}^{\infty} \cdots \sum_{x_K=0}^{\infty} \Phi(x_1, \ldots, x_K) \rho_1^{x_1} \cdots \rho_K^{x_K}$$

- Generating function of the balance function,
  - contains the same information as $\Phi(x)$.

- Performance measures can be derived from the normalization constant.

- Flow k throughput: $\gamma_k = \dfrac{\rho_k}{\mathrm{E}[x_k]} = \dfrac{G(\rho)}{\frac{\partial}{\partial \rho_k} G(\rho)} = \dfrac{1}{\frac{\partial}{\partial \rho_k} \log G(\rho)}$

Recursive algorithm for the normalization constant

Definitions

- Let $\mathcal{I}_k = \{i_1, \ldots, i_k\}, \quad 1 \le i_1 < \ldots < i_k \le K.$

- Partial sets of states

$$\Omega_{\mathcal{I}_k} = \{x : x_j > 0 \text{ if and only if } j \in \mathcal{I}_k\}$$

- $\mathcal{I}_0$ means the empty set $\emptyset$ and

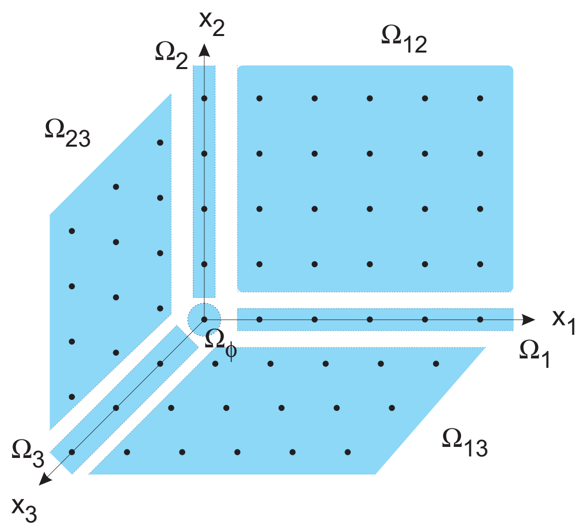
$$\Omega_\emptyset = \{(0, \ldots, 0)\}$$

Recursive algorithm – state space decomposition

- The whole state space is decomposed as

$$\Omega = \sum_{k=0}^{K} \sum_{\mathcal{I}_k} \Omega_{\mathcal{I}_k}.$$

- For instance $\Omega = \Omega_\emptyset + \Omega_1 + \Omega_2 + \Omega_3 + \Omega_{1,2} + \Omega_{1,3} + \Omega_{2,3} + \Omega_{1,2,3}$



Recursive algorithm – state space decomposition

- Partial sums over the sets $\Omega_{\mathcal{I}_k}$

$$G_{\mathcal{I}_k} = \sum_{x \in \Omega_{\mathcal{I}_k}} \Phi(x_1, \ldots, x_n) \rho_1^{x_1} \cdots \rho_n^{x_n}$$

- The normalization constant $G(\rho)$ is decomposed as

$$G(\rho) = \sum_{k=0}^{K} \sum_{\mathcal{I}_k} G_{\mathcal{I}_k}(\rho)$$

Recursion

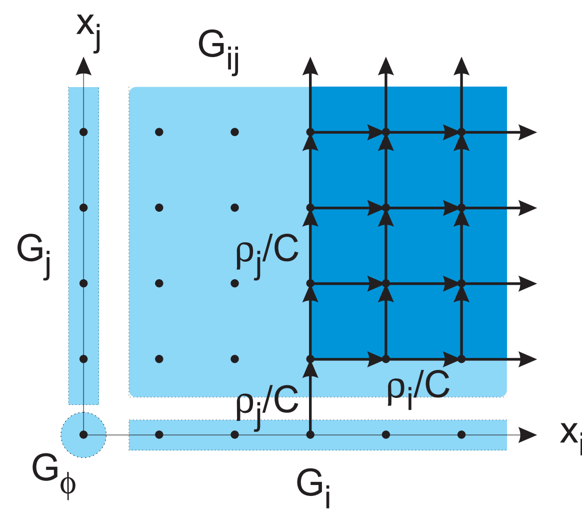
- Assumption: for each $\mathcal{I}_k$ a given link or links are saturated in all the states of $\Omega_{\mathcal{I}_k}$ (true for lines, trees,...).

- Under this assumption we can derive a recursion expressing $G_{\mathcal{I}_k}(\rho)$ in terms of the $G_{\mathcal{I}_{k-1}}(\rho)$.

- For a reasonable number of flow classes the number of different sets $\mathcal{I}_k$ is manageable (2^k).

Recursion (continued . . .)

- Consider a 2-dimensional set $\Omega_{i,j}$:



- State x in the set Ω_i contributes as a "source of recursion" an amount

$$\frac{\rho_j}{C} \cdot \frac{1}{1 - (\frac{\rho_i}{C} + \frac{\rho_j}{C})}$$

times its own measure $\pi(x)$.

Recursion (continued . . .)

- The first factor comes from the “bridge”.
- The second factor from the infinite sum S over the area in dark blue:

$$\sum_{x_1=0}^{\infty} \cdots \sum_{x_k=0}^{\infty} \binom{x_1+\cdots+x_k}{x_1,\ldots,x_k} y_{i_1}^{x_1} \cdots y_{i_k}^{x_k} = \sum_{x=0}^{\infty} \sum_{x_1+\cdots+x_k=x} \binom{x_1+\cdots+x_k}{x_1,\ldots,x_k} y_{i_1}^{x_1} \cdots y_{i_k}^{x_k}$$

$$= \sum_{x=0}^{\infty} (y_{i_1}+\cdots+y_{i_k})^x$$

$$= \frac{1}{1-(y_{i_1}+\cdots+y_{i_k})}$$

- Alternatively: $S = 1 + y_{i_1} S + \cdots + y_{i_k} S$

Recursion (continued . . .)

- Since the contribution is the same for each state in Ω_i, in total the set Ω_i gives to $G_{i,j}(\rho)$ a contribution

$$\frac{\rho_j}{C} \cdot \frac{1}{1 - (\frac{\rho_i}{C} + \frac{\rho_j}{C})} \cdot G_i(\rho)$$

- Similar contribution comes from the set Ω_j and the sought for recursion is

$$G_{i,j}(\rho) = \frac{\rho_j G_i(\rho) + \rho_i G_j(\rho)}{C - (\rho_i + \rho_j)}$$

General recursion for the normalization constant

- In general

$$G_{\mathcal{I}}(\rho) = \frac{\sum_{j \in \mathcal{I}'} \rho_j G_{\mathcal{I} \setminus j}(\rho)}{C_{\sigma(\mathcal{I})} - \sum_{j \in \mathcal{I}'} \rho_j}$$

  $\sigma(\mathcal{I})$ is the link which is saturated when $x_j > 0$ iff $j \in \mathcal{I}$

  $\mathcal{I}' \subset \mathcal{I}$ stands for those classes $j \in \mathcal{I}$ for which $\sigma(\mathcal{I}) \in r_j$

- If $\sigma(\mathcal{I})$ is not unique any of the saturated links can be used as the basis for the recursion.

Recursion for the throughput

- Direct recursion for the throughput:

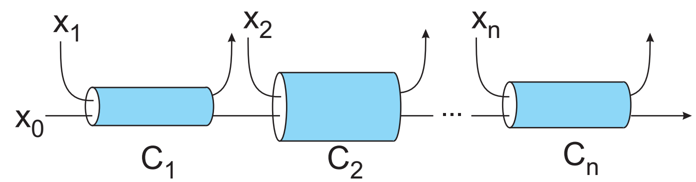
$$\gamma_i = \frac{G(\rho)}{\frac{\partial}{\partial \rho_i} G(\rho)}$$

- The denominator is decomposed as

$$\frac{\partial}{\partial \rho_i} G(\rho) = \sum_{k=0}^{K} \sum_{\mathcal{I}_k} \frac{\partial}{\partial \rho_i} G_{\mathcal{I}_k}(\rho) \equiv \sum_{k=0}^{K} \sum_{\mathcal{I}_k} H^{(i)}_{\mathcal{I}_k}(\rho)$$

$$H^{(i)}_{\mathcal{I}_k}(\rho) = \frac{1_{i \in \mathcal{I}'_k}\left(G_{\mathcal{I}_k}(\rho) + G_{\mathcal{I}^{(i)}_k}(\rho)\right) + \sum_{j \in \mathcal{I}'_k} \rho_j H^{(i)}_{\mathcal{I}^{(j)}_k}(\rho)}{C_{\sigma(\mathcal{I}_k)} - \sum_{j \in \mathcal{I}'_k} \rho_j}$$

Examples: Inhomogeneous line



- Redefine $\Omega_i = \{x : x_j = 0 \text{ for } j > i\}$
- Then the recursion reads

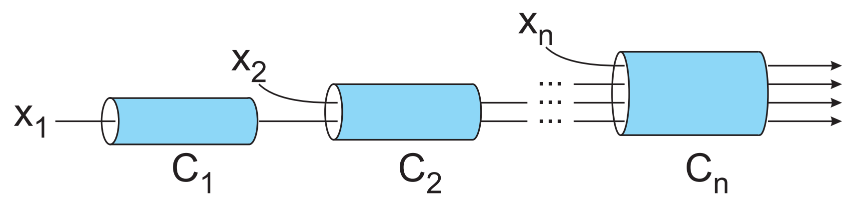
$$\begin{cases} G_0(\rho) & = \frac{1}{1-\frac{\rho_0}{C}} \\ G_i(\rho) & = \left(1 + \frac{\frac{\rho_i}{C_i}}{1-\frac{\rho_0+\rho_i}{C_i}}\right) \cdot G_{i-1}(\rho) \;=\; \frac{1-\frac{\rho_0}{C_i}}{1-\frac{\rho_0+\rho_i}{C_i}} \cdot G_{i-1}(\rho) \end{cases}$$

$$G(\rho) = \frac{1}{1-\frac{\rho_0}{C}} \cdot \frac{1-\frac{\rho_0}{C_1}}{1-\frac{\rho_0+\rho_1}{C_1}} \cdots \frac{1-\frac{\rho_0}{C_n}}{1-\frac{\rho_0+\rho_n}{C_n}}$$

- The throughput

$$\gamma_0 = \left(\frac{1}{C-\rho_0} + \sum_{l=1}^{n}\left(\frac{1}{C_l-\rho_l-\rho_0} - \frac{1}{C_l-\rho_0}\right)\right)^{-1}$$

Examples: Parking lot



- The recursion reads

$$\begin{cases} G_1(\rho) &= \frac{1}{1-\frac{\rho_1}{C_1}} \\ G_i(\rho) &= \left(1 + \frac{\frac{\rho_i}{C_i}}{1-\frac{\rho_1+\cdots+\rho_i}{C_i}}\right) \cdot G_{i-1}(\rho) = \frac{1-\frac{\rho_1+\cdots+\rho_{i-1}}{C_i}}{1-\frac{\rho_1+\cdots+\rho_i}{C_i}} \cdot G_{i-1}(\rho) \end{cases}$$

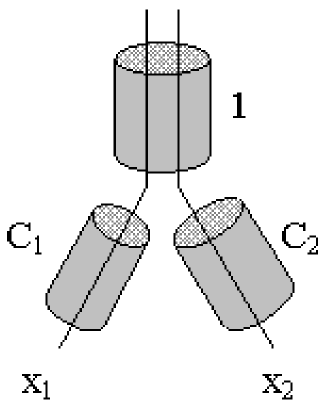
- By denoting the link i load by $R_i = \sum_{j=1}^{i} \rho_j$

$$G(\rho) = \frac{1}{1-\frac{R_1}{C_1}} \cdot \frac{1-\frac{R_1}{C_2}}{1-\frac{R_2}{C_2}} \cdots \frac{1-\frac{R_{n-1}}{C_n}}{1-\frac{R_n}{C_n}}$$

- The throughput

$$\gamma_i = \left(\frac{1}{C_i-R_i} + \sum_{l=i+1}^{n}\left(\frac{1}{C_l-R_l} - \frac{1}{C_l-R_{l-1}}\right)\right)^{-1}$$

Examples: 2-branch tree



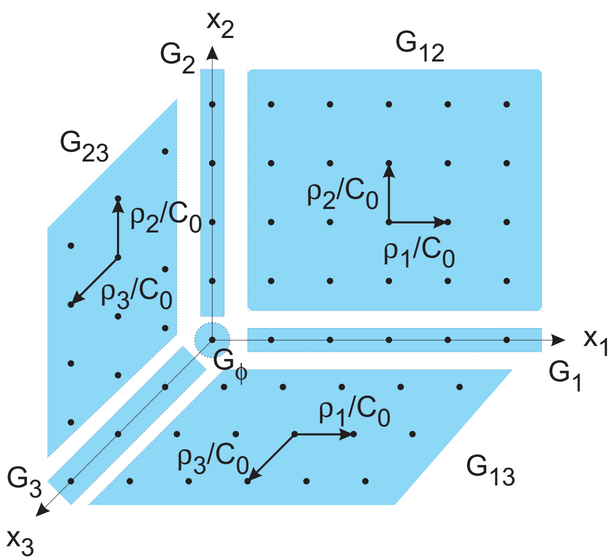
$$\begin{cases} G_\emptyset(\rho) & = & 1 \\ G_1(\rho) & = & \dfrac{\frac{\rho_1}{C_1} \cdot G_\emptyset(\rho)}{1 - \frac{\rho_1}{C_1}} \\ G_2(\rho) & = & \dfrac{\frac{\rho_2}{C_2} \cdot G_\emptyset(\rho)}{1 - \frac{\rho_2}{C_2}} \\ G_{1,2}(\rho) & = & \dfrac{\frac{\rho_1}{C_0} \cdot G_2(\rho) + \frac{\rho_2}{C_0} \cdot G_1(\rho)}{1 - \frac{\rho_1+\rho_2}{C_0}} \end{cases}$$

$$G(\rho) = \frac{1}{1 - \frac{\rho_1+\rho_2}{C_0}} \cdot \left(\frac{1 - \frac{\rho_1}{C_0}}{1 - \frac{\rho_1}{C_1}} + \frac{1 - \frac{\rho_2}{C_0}}{1 - \frac{\rho_2}{C_2}} - 1 \right)$$

$$= \frac{1 - \frac{\rho_1+\rho_2}{C_0} + \frac{\rho_1}{C_1}\frac{\rho_2}{C_2}(\frac{C_1+C_2}{C_0} - 1)}{(1 - \frac{\rho_1+\rho_2}{C_0})(1 - \frac{\rho_1}{C_1})(1 - \frac{\rho_2}{C_2})}$$

Examples: 3-branch tree

Case 1. All pairs add to more than 1, e.g., $C_1 = 0.9$, $C_2 = 0.6$, $C_3 = 0.5$



$$\begin{cases} G_\emptyset(\rho) &= 1 \\ G_1(\rho) &= \dfrac{\frac{\rho_1}{C_1}\cdot G_\emptyset(\rho)}{1-\frac{\rho_1}{C_1}} \\ G_2(\rho) &= \dfrac{\frac{\rho_2}{C_2}\cdot G_\emptyset(\rho)}{1-\frac{\rho_2}{C_2}} \\ G_3(\rho) &= \dfrac{\frac{\rho_3}{C_3}\cdot G_\emptyset(\rho)}{1-\frac{\rho_3}{C_3}} \\ G_{1,2}(\rho) &= \dfrac{\frac{\rho_1}{C_0}\cdot G_2(\rho)+\frac{\rho_2}{C_0}\cdot G_1(\rho)}{1-\frac{\rho_1+\rho_2}{C_0}} \\ G_{1,3}(\rho) &= \dfrac{\frac{\rho_1}{C_0}\cdot G_3(\rho)+\frac{\rho_3}{C_0}\cdot G_1(\rho)}{1-\frac{\rho_1+\rho_3}{C_0}} \\ G_{2,3}(\rho) &= \dfrac{\frac{\rho_2}{C_0}\cdot G_3(\rho)+\frac{\rho_3}{C_0}\cdot G_2(\rho)}{1-\frac{\rho_2+\rho_3}{C_0}} \\ G_{1,2,3}(\rho) &= \dfrac{\frac{\rho_1}{C_0}\cdot G_{2,3}(\rho)+\frac{\rho_2}{C_0}\cdot G_{1,3}(\rho)+\frac{\rho_3}{C_0}\cdot G_{1,2}(\rho)}{1-\frac{\rho_1+\rho_2+\rho_3}{C_0}} \end{cases}$$

Examples: 3-branch tree

Case 1. All pairs add to more than 1, e.g., $C_1 = 0.9$, $C_2 = 0.6$, $C_3 = 0.5$

The sum of these can be simplified to

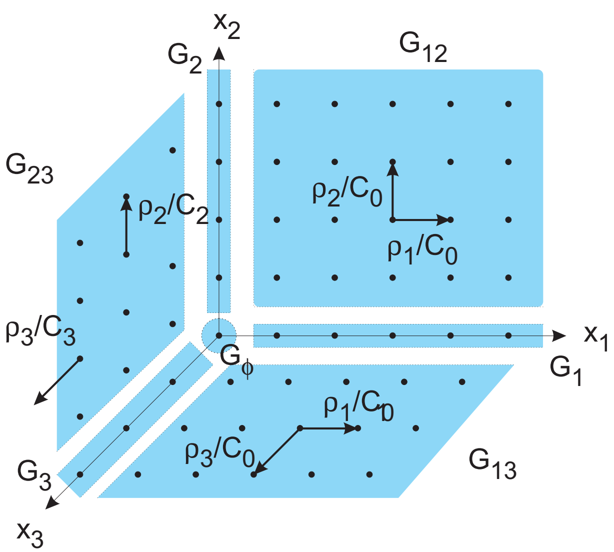
$$G(\rho) = \frac{1}{1 - \frac{\rho_1+\rho_2+\rho_3}{C_0}} \left(\frac{1 - \frac{\rho_1}{C_0}}{1 - \frac{\rho_1}{C_1}} + \frac{1 - \frac{\rho_2}{C_0}}{1 - \frac{\rho_2}{C_2}} + \frac{1 - \frac{\rho_3}{C_0}}{1 - \frac{\rho_3}{C_3}} - 2 \right)$$

$$= \frac{1 - \frac{\rho_1+\rho_2+\rho_3}{C_0} + \frac{\rho_1}{C_1}\frac{\rho_2}{C_2}(\frac{C_1+C_2}{C_0} - 1) + \frac{\rho_2}{C_2}\frac{\rho_3}{C_3}(\frac{C_2+C_3}{C_0} - 1) + \frac{\rho_3}{C_3}\frac{\rho_1}{C_1}(\frac{C_3+C_1}{C_0} - 1) + \frac{\rho_1}{C_1}\frac{\rho_2}{C_2}\frac{\rho_3}{C_3}(2 - \frac{C_1+C_2+C_3}{C_0})}{(1 - \frac{\rho_1+\rho_2+\rho_3}{C_0})(1 - \frac{\rho_1}{C_1})(1 - \frac{\rho_2}{C_2})(1 - \frac{\rho_3}{C_3})}$$

with an obvious generalization to any number of branches.

Examples: 3-branch tree

Case 2. Two pairs add to more than 1, e.g., $C_1 = 0.9$, $C_2 = 0.6$, $C_3 = 0.2$



$$\begin{cases} G_\emptyset(\rho) &= 1 \\ G_1(\rho) &= \dfrac{\frac{\rho_1}{C_1}\cdot G_\emptyset(\rho)}{1-\frac{\rho_1}{C_1}} \\ G_2(\rho) &= \dfrac{\frac{\rho_2}{C_2}\cdot G_\emptyset(\rho)}{1-\frac{\rho_2}{C_2}} \\ G_3(\rho) &= \dfrac{\frac{\rho_3}{C_3}\cdot G_\emptyset(\rho)}{1-\frac{\rho_3}{C_3}} \\ G_{1,2}(\rho) &= \dfrac{\frac{\rho_1}{C_0}\cdot G_2(\rho)+\frac{\rho_2}{C_0}\cdot G_1(\rho)}{1-\frac{\rho_1+\rho_2}{C_0}} \\ G_{1,3}(\rho) &= \dfrac{\frac{\rho_1}{C_0}\cdot G_3(\rho)+\frac{\rho_3}{C_0}\cdot G_1(\rho)}{1-\frac{\rho_1+\rho_3}{C_0}} \\ G_{2,3}(\rho) &= \dfrac{\frac{\rho_2}{C_2}\cdot G_3(\rho)}{1-\frac{\rho_2}{C_2}} = \dfrac{\frac{\rho_3}{C_3}\cdot G_2(\rho)}{1-\frac{\rho_3}{C_3}} \\ G_{1,2,3}(\rho) &= \dfrac{\frac{\rho_1}{C_0}\cdot G_{2,3}(\rho)+\frac{\rho_2}{C_0}\cdot G_{1,3}(\rho)+\frac{\rho_3}{C_0}\cdot G_{1,2}(\rho)}{1-\frac{\rho_1+\rho_2+\rho_3}{C_0}} \end{cases}$$

Examples: 3-branch tree

Case 2. Two pairs add to more than 1, e.g., $C_1 = 0.9$, $C_2 = 0.6$, $C_3 = 0.2$

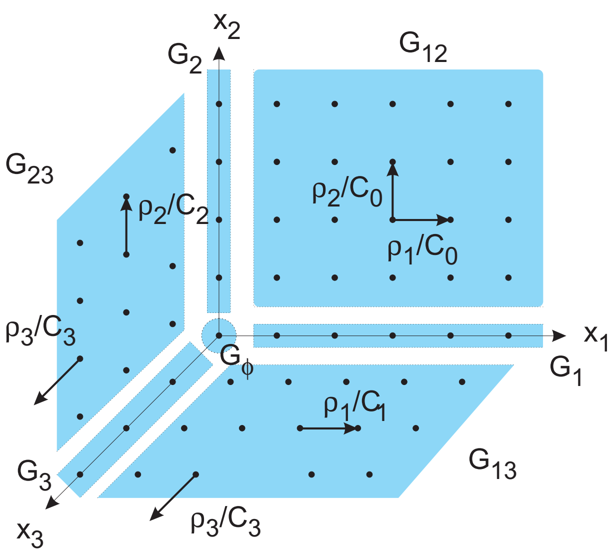
After simplification

$$G(\rho) = \frac{1}{1-\frac{\rho_1+\rho_2+\rho_3}{C_0}}\left(\frac{1-\frac{\rho_1}{C_0}}{1-\frac{\rho_1}{C_1}} + \frac{1-\frac{\rho_2+\rho_3}{C_0}}{(1-\frac{\rho_2}{C_2})(1-\frac{\rho_3}{C_3})} - 1\right)$$

$$= \frac{1-\frac{\rho_1+\rho_2+\rho_3}{C_0} + \frac{\rho_1}{C_1}\frac{\rho_2}{C_2}(\frac{C_1+C_2}{C_0}-1) + \frac{\rho_1}{C_1}\frac{\rho_3}{C_3}(\frac{C_1+C_3}{C_0}-1) + \frac{\rho_1}{C_1}\frac{\rho_2}{C_2}\frac{\rho_3}{C_3}(1-\frac{C_1}{C_0})}{(1-\frac{\rho_1+\rho_2+\rho_3}{C_0})(1-\frac{\rho_1}{C_1})(1-\frac{\rho_2}{C_2})(1-\frac{\rho_3}{C_3})}$$

Examples: 3-branch tree

Case 3. One pair adds to more than 1, e.g., $C_1 = 0.7$, $C_2 = 0.4$, $C_3 = 0.2$



$$
\begin{cases}
G_\emptyset(\rho) = 1 \\
G_1(\rho) = \dfrac{\frac{\rho_1}{C_1} \cdot G_\emptyset(\rho)}{1 - \frac{\rho_1}{C_1}} \\
G_2(\rho) = \dfrac{\frac{\rho_2}{C_2} \cdot G_\emptyset(\rho)}{1 - \frac{\rho_2}{C_2}} \\
G_3(\rho) = \dfrac{\frac{\rho_3}{C_3} \cdot G_\emptyset(\rho)}{1 - \frac{\rho_3}{C_3}} \\
G_{1,2}(\rho) = \dfrac{\frac{\rho_1}{C_0} \cdot G_2(\rho) + \frac{\rho_2}{C_0} \cdot G_1(\rho)}{1 - \frac{\rho_1 + \rho_2}{C_0}} \\
G_{1,3}(\rho) = \dfrac{\frac{\rho_1}{C_1} \cdot G_3(\rho)}{1 - \frac{\rho_1}{C_1}} = \dfrac{\frac{\rho_3}{C_3} \cdot G_1(\rho)}{1 - \frac{\rho_3}{C_3}} \\
G_{2,3}(\rho) = \dfrac{\frac{\rho_2}{C_2} \cdot G_3(\rho)}{1 - \frac{\rho_2}{C_2}} = \dfrac{\frac{\rho_3}{C_3} \cdot G_2(\rho)}{1 - \frac{\rho_3}{C_3}} \\
G_{1,2,3}(\rho) = \dfrac{\frac{\rho_1}{C_0} \cdot G_{2,3}(\rho) + \frac{\rho_2}{C_0} \cdot G_{1,3}(\rho) + \frac{\rho_3}{C_0} \cdot G_{1,2}(\rho)}{1 - \frac{\rho_1 + \rho_2 + \rho_3}{C_0}}
\end{cases}
$$

Examples: 3-branch tree

Case 3. One pair adds to more than 1, e.g., $C_1 = 0.7$, $C_2 = 0.4$, $C_3 = 0.2$

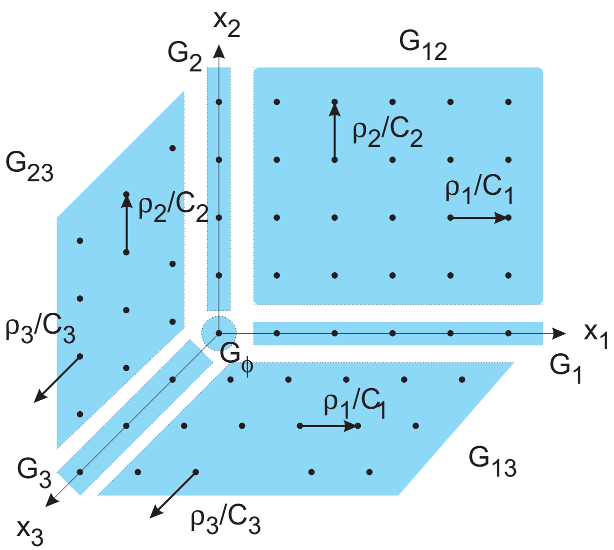
After simplification

$$G(\rho) = \frac{1}{1-\frac{\rho_1+\rho_2+\rho_3}{C_0}} \cdot \frac{1}{1-\frac{\rho_3}{C_3}} \left(\frac{1-\frac{\rho_1+\rho_3}{C_0}}{1-\frac{\rho_1}{C_1}} + \frac{1-\frac{\rho_2+\rho_3}{C_0}}{1-\frac{\rho_2}{C_2}} - (1-\frac{\rho_3}{C_0}) \right)$$

$$= \frac{1-\frac{\rho_1+\rho_2+\rho_3}{C_0} + \frac{\rho_1}{C_1}\frac{\rho_2}{C_2}(\frac{C_1+C_2}{C_0}-1) + \frac{\rho_1}{C_1}\frac{\rho_2}{C_2}\frac{\rho_3}{C_3}\frac{C_3}{C_0}}{(1-\frac{\rho_1+\rho_2+\rho_3}{C_0})(1-\frac{\rho_1}{C_1})(1-\frac{\rho_2}{C_2})(1-\frac{\rho_3}{C_3})}$$

Examples: 3-branch tree

Case 4. None of the pairs adds to more than 1, e.g., $C_1 = 0.5$, $C_2 = 0.4$, $C_3 = 0.2$



$$\begin{cases} G_\emptyset(\rho) &= 1 \\ G_1(\rho) &= \dfrac{\frac{\rho_1}{C_1}\cdot G_\emptyset(\rho)}{1-\frac{\rho_1}{C_1}} \\ G_2(\rho) &= \dfrac{\frac{\rho_2}{C_2}\cdot G_\emptyset(\rho)}{1-\frac{\rho_2}{C_2}} \\ G_3(\rho) &= \dfrac{\frac{\rho_3}{C_3}\cdot G_\emptyset(\rho)}{1-\frac{\rho_3}{C_3}} \\ G_{1,2}(\rho) &= \dfrac{\frac{\rho_1}{C_1}\cdot G_2(\rho)}{1-\frac{\rho_1}{C_1}} = \dfrac{\frac{\rho_2}{C_2}\cdot G_1(\rho)}{1-\frac{\rho_2}{C_2}} \\ G_{1,3}(\rho) &= \dfrac{\frac{\rho_1}{C_1}\cdot G_3(\rho)}{1-\frac{\rho_1}{C_1}} = \dfrac{\frac{\rho_3}{C_3}\cdot G_1(\rho)}{1-\frac{\rho_3}{C_3}} \\ G_{2,3}(\rho) &= \dfrac{\frac{\rho_2}{C_2}\cdot G_3(\rho)}{1-\frac{\rho_2}{C_2}} = \dfrac{\frac{\rho_3}{C_3}\cdot G_2(\rho)}{1-\frac{\rho_3}{C_3}} \\ G_{1,2,3}(\rho) &= \dfrac{\frac{\rho_1}{C_0}\cdot G_{2,3}(\rho)+\frac{\rho_2}{C_0}\cdot G_{1,3}(\rho)+\frac{\rho_3}{C_0}\cdot G_{1,2}(\rho)}{1-\frac{\rho_1+\rho_2+\rho_3}{C_0}} \end{cases}$$

Examples: 3-branch tree

Case 4. None of the pairs adds to more than 1, e.g., $C_1 = 0.5$, $C_2 = 0.4$, $C_3 = 0.2$

After simplification

$$G(\rho) = \frac{1 - \frac{\rho_1+\rho_2+\rho_3}{C_0} + \frac{\rho_1}{C_1}\frac{\rho_2}{C_2}\frac{\rho_3}{C_3}\left(\frac{C_1+C_2+C_3}{C_0} - 1\right)}{\left(1 - \frac{\rho_1+\rho_2+\rho_3}{C_0}\right)\left(1 - \frac{\rho_1}{C_1}\right)\left(1 - \frac{\rho_2}{C_2}\right)\left(1 - \frac{\rho_3}{C_3}\right)}$$

Examples: Homogeneous n-branch tree

- All branches have equal capacity and equal load, $C_i = C,\ \rho_i = \rho$. The capacity of the trunk is C_0.

- Denote by m the largest integer such that $m \times C < C_0$.

- Applying the recursion on any set of states where given k branches are saturated we have

$$G_k(\rho) = \left(\frac{\rho}{C - \rho}\right)^k, \quad k = 0, \ldots, m$$

- There are $\binom{n}{k}$ sets of this type for any k.

Examples: Homogeneous n-branch tree (continued . . .)

- Applying the recursion to the set $\hat{\Omega}$ where the trunk is saturated, we get

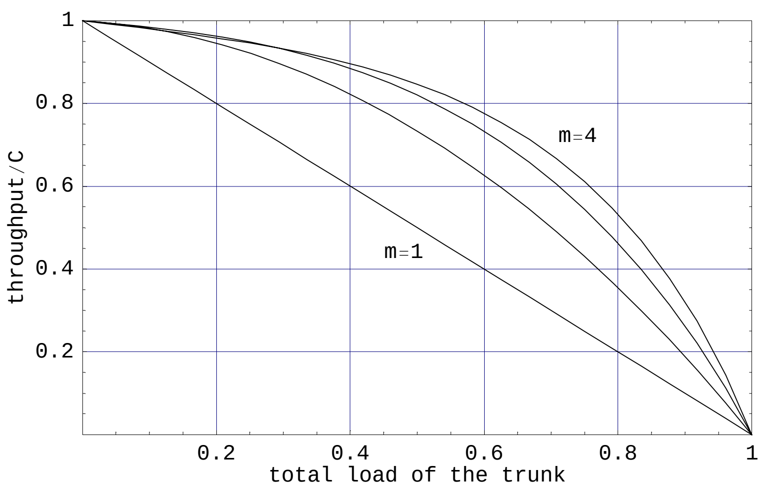
$$\hat{G}(\rho) = \frac{(n-m)\rho}{C_0 - n\rho} \binom{n}{m} G_m$$

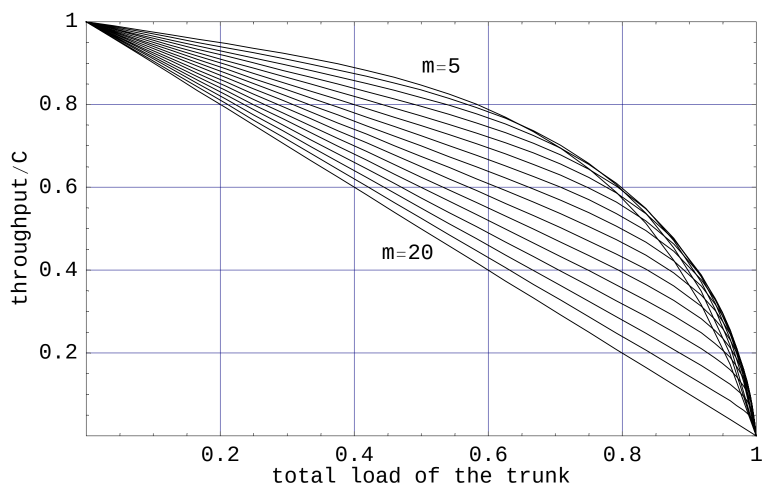
- Thus

$$G(\rho) = \sum_{k=0}^{m} \binom{n}{k} \left(\frac{\rho}{C-\rho} \right)^k + \frac{(n-m)\rho}{C_0 - n\rho} \binom{n}{m} \left(\frac{\rho}{C-\rho} \right)^m$$

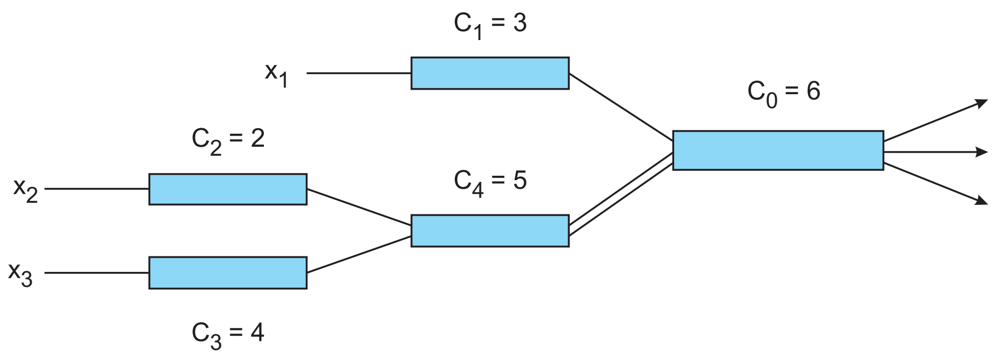
Examples: Homogeneous n-branch tree (continued . . .)

- $n = 20$ branches, $m = 1, \ldots, 20$





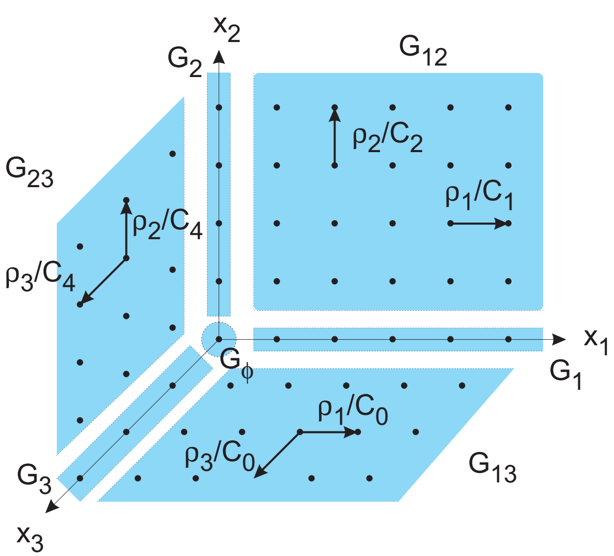
Examples: 3-level tree



With the capacities given in the figure one infers

$$\begin{cases} \mathcal{I}' = \{1\} \text{ or } \{2\} & \text{with links 1 and 2 saturated} & \text{for } \mathcal{I} = \{1,2\} \\ \mathcal{I}' = \{1,3\} & \text{with link 0 saturated} & \text{for } \mathcal{I} = \{1,3\} \\ \mathcal{I}' = \{2,3\} & \text{with link 4 saturated} & \text{for } \mathcal{I} = \{2,3\} \\ \mathcal{I}' = \{1,2,3\} & \text{with link 0 saturated} & \text{for } \mathcal{I} = \{1,2,3\} \end{cases}$$

Examples: 3-level tree (continued . . .)



$$
\begin{cases}
G_\emptyset(\rho) = 1 \\
G_1(\rho) = \frac{\frac{\rho_1}{C_1}\cdot G_\emptyset(\rho)}{1-\frac{\rho_1}{C_1}} \\
G_2(\rho) = \frac{\frac{\rho_2}{C_2}\cdot G_\emptyset(\rho)}{1-\frac{\rho_2}{C_2}} \\
G_3(\rho) = \frac{\frac{\rho_3}{C_3}\cdot G_\emptyset(\rho)}{1-\frac{\rho_3}{C_3}} \\
G_{1,2}(\rho) = \frac{\frac{\rho_1}{C_1}\cdot G_2(\rho)}{1-\frac{\rho_1}{C_1}} = \frac{\frac{\rho_2}{C_2}\cdot G_1(\rho)}{1-\frac{\rho_2}{C_2}} \\
G_{1,3}(\rho) = \frac{\frac{\rho_1}{C_0}\cdot G_3(\rho)+\frac{\rho_3}{C_0}\cdot G_1(\rho)}{1-\frac{\rho_1+\rho_3}{C_0}} \\
G_{2,3}(\rho) = \frac{\frac{\rho_2}{C_4}\cdot G_3(\rho)+\frac{\rho_3}{C_4}\cdot G_2(\rho)}{1-\frac{\rho_2+\rho_3}{C_4}} \\
G_{1,2,3}(\rho) = \frac{\frac{\rho_1}{C_0}\cdot G_{2,3}(\rho)+\frac{\rho_2}{C_0}\cdot G_{1,3}(\rho)+\frac{\rho_3}{C_0}\cdot G_{1,2}(\rho)}{1-\frac{\rho_1+\rho_2+\rho_3}{C_0}}
\end{cases}
$$

Examples: 3-level tree (continued . . .)

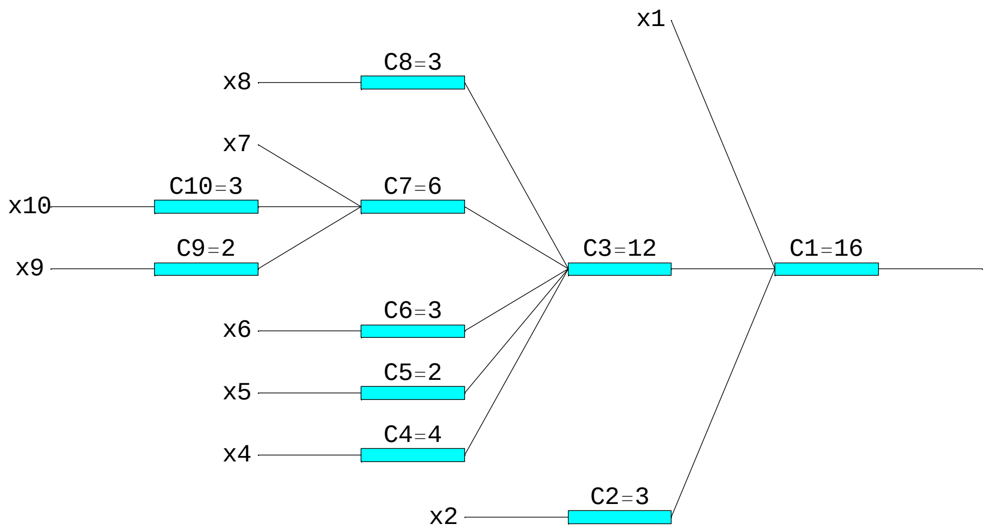
For the capacities given in the figure one gets with the aid of Mathematica

$$G(\rho) = \frac{720 - 120\,\rho_1 - 264\,(\rho_2 + \rho_3) + 24\,(\rho_1\rho_2 + {\rho_2}^2 + {\rho_3}^2) + 3\,\rho_2\rho_3(22 - (\rho_2 + \rho_3)) + \rho_1\rho_3(34 + (2 + \rho_2)(\rho_2 - \rho_3))}{(3 - \rho_1)\,(2 - \rho_2)\,(4 - \rho_3)\,(5 - \rho_2 - \rho_3)\,(6 - \rho_1 - \rho_2 - \rho_3)}$$

As an example of the throughputs, with $\rho_2 = \rho_3 = 1$, one gets

$$\begin{cases} \gamma_1 = \dfrac{(4 - \rho_1)\,(3 - \rho_1)\,(75 - 16\,\rho_1)}{333 - 150\,\rho_1 + 16\,{\rho_1}^2} \\ \gamma_2 = \dfrac{12\,(4 - \rho_1)\,(75 - 16\,\rho_1)}{3792 - 1663\,\rho_1 + 187\,{\rho_1}^2} \\ \gamma_3 = \dfrac{12\,(4 - \rho_1)\,(75 - 16\,\rho_1)}{1392 - 479\,\rho_1 + 41\,{\rho_1}^2} \end{cases}$$

Examples: A larger tree



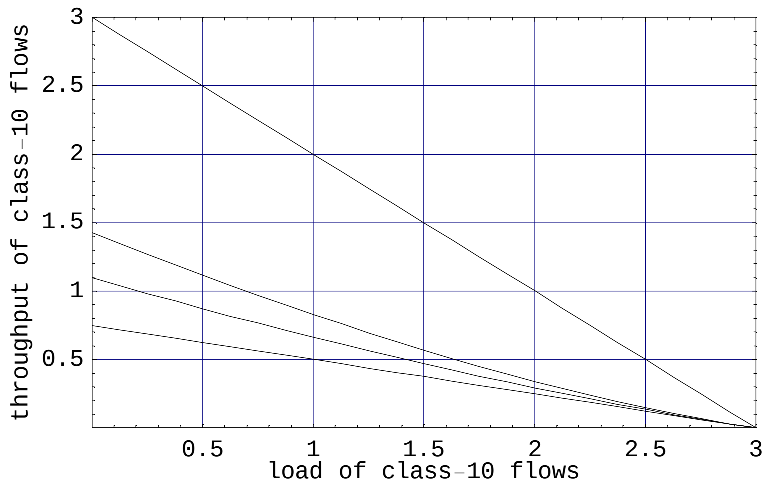
- We study the throughput of flow 10 going through links 1, 3, 6, and 10 as a function of its own load ρ_{10}.
- The other classes are assumed to have fixed loads as follows: $\rho_1 = \rho_2 = \rho_4 = \rho_6 = \rho_7 = 2$ and $\rho_5 = \rho_8 = \rho_9 = 1$.
- With these loads all the four links on the route of flow 10 have the average residual capacity of 3 units.

Examples: A larger tree (continued . . .)

- As before, one derives the normalization constant

$$G(\rho_{10}) = \frac{6\,(5-\rho_{10})\left(951-411\,\rho_{10}+46\,\rho_{10}{}^2\right)}{(3-\rho_{10})^4}$$

- Comparison of the exact throughput with simple bounds:



From bottom up: store-and-forward, parking lot, exact, deterministic.

Limited access rates

- A single link with capacity C offered flows from K different classes.

- All flows in each class i are limited by an access rate a_i.

- The capacity constraints must then be supplemented by the requirement $\phi_i(x) \leq x_i a_i$ for all x and i and the basic recursion has to be modified accordingly.

- As long as the sum of the access rates of the active flows is less than C each flow is allocated a rate equal to its access rate. Only when the sum exceeds C becomes the capacity sharing effective.

Limited access rates (continued . . .)

- For convenience think in terms of integer valued rates a_i and capacity C.

- Denote $\Omega_\emptyset = \{x : x \cdot a \leq C\}$ and $G_\emptyset = \sum_{x \in \Omega_\emptyset} \pi(x)$.

- The iman-Roberts recursion can be used to obtain

$$G_\emptyset = \sum_{c=0}^{C} g(c), \qquad g(c) = \sum_{k=0}^{K} \frac{\rho_k}{c} g(c - a_k), \qquad c = 1, 2, \ldots$$

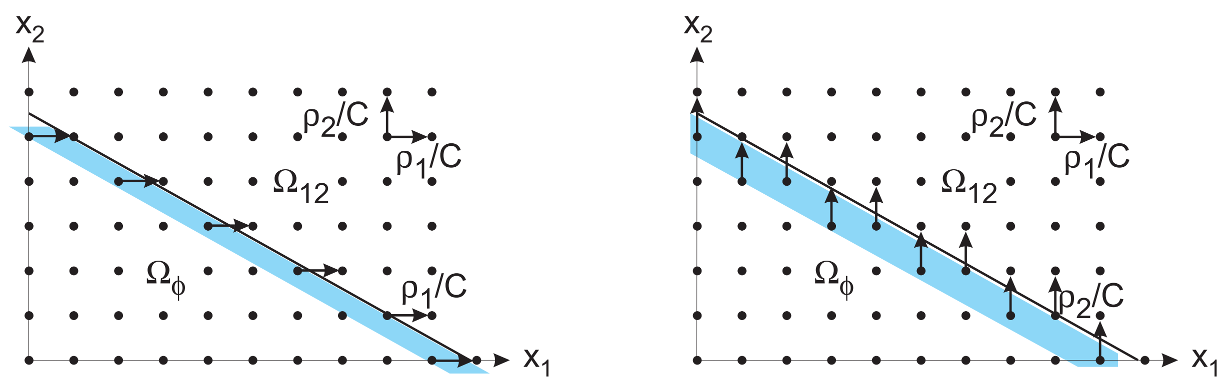
with $g(1) = 1$ and $g(c) = 0$ for $c < 0$.

Limited access rates (continued . . .)

- In addition it gives the measures of the "blocking sets" (referring to a virtual multi-bitrate system, where the flows are inelastic having fixed bandwidth requirements of a_k),

$$B_k = \sum_{c=C-a_k+1}^{C} g(c)$$

- The blocking sets form the boundary between inelastic and elastic capacity sharing:



Limited access rates (continued ...)

- As before, each state x in the blocking set of class i contributes as a "source of recursion" an amount

$$\frac{\rho_i}{C} \cdot \frac{1}{1-\frac{\rho}{C}} \cdot \pi(x)$$

where $\rho = \sum_k \rho_k$ is the total load.

- Thus, the normalization constant is

$$G = \sum_{c=0}^{C} g(c) + \sum_{k=1}^{K} \frac{B_k \rho_k}{(C-\rho)}$$

Limited access rates (continued . . .)

- A similar idea can be applied also in the case of a tree with access rates.
- The algorithm becomes, however, more complicated.

$$G = \sum_{\mathcal{S} \in \Sigma(\mathcal{T})} G_{\mathcal{S}}$$

where $\mathcal{S}$ is a saturation set and $\Sigma(\mathcal{T})$ is the set of all feasible saturation sets in tree $\mathcal{T}$.

Limited access rates (continued . . .)

- General reduction:

$$G_{\mathcal{S}}[\mathcal{T}] = G_{\emptyset}[\mathcal{T} \mathop{\backslash\backslash}_{l \in \mathcal{S}} \mathcal{T}_l] \prod_{l \in \mathcal{S}} G_l[\mathcal{T}_l]$$

- Recursion for a subtree $\mathcal{T}_l$ with saturated root l

$$G_l[\mathcal{T}_l] = \frac{\displaystyle\sum_{\mathcal{S} \in \hat{\Sigma}(\mathcal{T}_l)} \prod_{j \in \mathcal{S}} G_j[\mathcal{T}_j] \sum_{i \in \mathcal{F}_l \backslash \cup_{j \in \mathcal{S}} \mathcal{F}_j} \rho_i G_{\emptyset|_i l}[\mathcal{T}_l \mathop{\backslash\backslash}_{j \in \mathcal{S}} \mathcal{T}_j]}{\displaystyle C_l - \sum_{i \in \mathcal{F}_l} \rho_i}$$

- Partial class-k blocking measure “due to the root link l”

$$G_{\emptyset|_i l}[\mathcal{T}_l(C)] = G_{\emptyset}[\mathcal{T}_l(C)] - G_{\emptyset}[\mathcal{T}_l(C - a_i r_i[\mathcal{T}_l])] - \sum_{j \in \hat{\mathcal{R}}_i[\mathcal{T}_l]} G_{\emptyset}[\mathcal{T}_l(C) \backslash\backslash \mathcal{T}_j(C)] \cdot G_{\emptyset|_i j}[\mathcal{T}_j(C)]$$

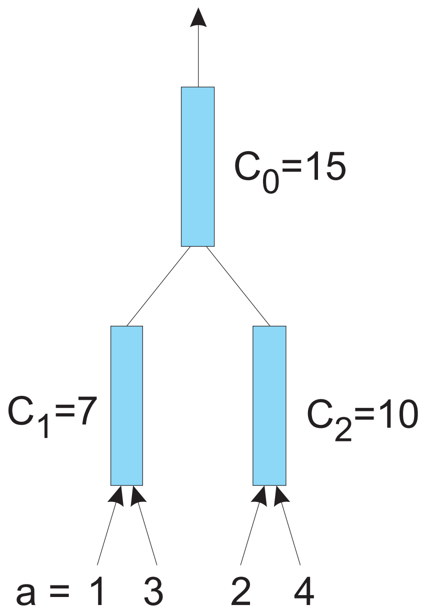
Limited access rates (continued . . .)

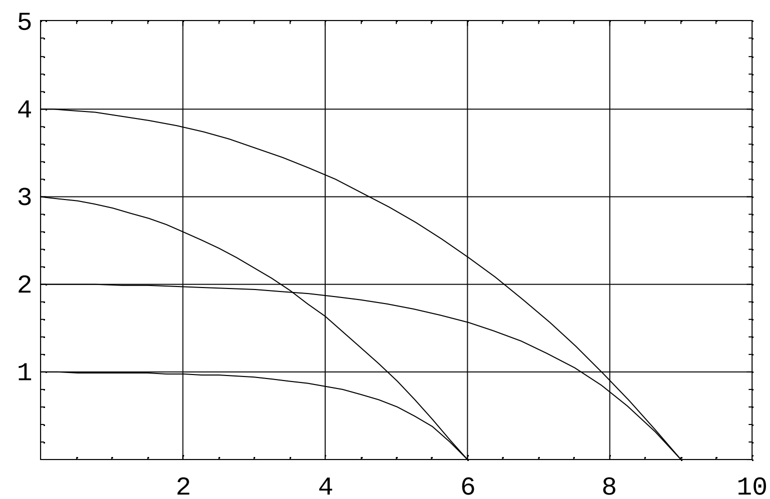
Above we have used the following notation:

$$
\begin{cases}
\mathcal{T} & = \text{a tree (set of links; a link is indexed by the set of flows going through it)} \\
\mathcal{T}(C) & = \text{a tree, emphasizing the capacities of its links } (C \text{ is the capacity vector)} \\
\mathcal{T}_l & = \text{subtree of } \mathcal{T} \text{ having link } l \text{ as the root (exception: the third bullet)} \\
G_{\mathcal{S}}[\mathcal{T}] & = \text{state sum over the state space } \Omega_{\mathcal{S}} \text{ of tree } \mathcal{T} \text{ with saturation set } \mathcal{S} \\
G_l[\mathcal{T}_l] & = \text{as the previous one, the saturation set comprising the root of tree } \mathcal{T}_l \text{ only} \\
G_{\emptyset|_i l}[\mathcal{T}_l] & = \text{state sum over the border set } \Omega_{\emptyset|_i l} \text{ of tree } \mathcal{T}_l \\
\hat{\mathcal{R}}_i[\mathcal{T}_l] & = \text{route of flow class } i \text{ in tree } \mathcal{T}_l \text{ excluding the root } l \\
r_i^{\mathcal{T}} & = \text{a vector of the same form as the capacity vector of tree } \mathcal{T}\text{, with 1 in each entry} \\
 & \text{corresponding to a link used by route } \mathcal{R}_i^{\mathcal{T}} \text{ and 0 elsewhere} \\
\mathcal{T} \backslash\backslash \mathcal{T}_l & = \text{tree remaining when subtree } \mathcal{T}_l \text{ is detached from tree } \mathcal{T} \text{ and the capacity } C_l \text{ is} \\
 & \text{subtracted from the capacitites of the links on the route from } l \text{ to the root of } \mathcal{T} \\
\mathcal{T} \mathop{\backslash\backslash}_{l \in \mathcal{S}} \mathcal{T}_l & = \text{the above subtraction operation repeated over all subtrees } \mathcal{T}_l \text{ with } l \in \mathcal{S}
\end{cases}
$$

Limited access rates (continued . . .)

Example





- The graph shows the throughput of each flow class as a function of its own load assuming all the other loads equalling 1.

Closed networks

- Reducible routing: the nodes are partitioned into K subsets of nodes such that routing is irreducible on each subset c_k. M_k flows in the routing group k.

- Node i_k is the "source" of set c_k: the capacity allocation to this node, $\psi_k(x_{i_k})$, is a function of x_{i_k} only. Its load is denoted ϱ_k.

- The arrival frequencies λ_i to different nodes are determined up to a multiplicative constant by

$$\lambda_i = \sum_{j \in c_k} \lambda_j p_{ji}, \qquad i \in c_k.$$

Closed networks (continued)

- The invariant measure is

$$\pi(x) = \begin{cases} \Phi(x') \prod_{k=1}^{K} \Psi_k(x_{i_k}) \prod_{i=1}^{N} \rho_i^{x_i}, & \text{for } \sum_{i \in c_k} x_i = M_k,\ \forall k \\ 0, & \text{otherwise} \end{cases}$$

where x' is the state vector excluding the source nodes, $\Phi(x')$ is the balance function for those nodes, and $\Psi_k(n) = 1/\psi_k(1) \cdots \psi_k(n)$.

- The state sum with $M_1, \ldots, M_K$ flows in each class is denoted

$$G_{M_1,\ldots,M_K}(\rho) = \sum_{\sum_{i \in c_k} x_i = M_k,\ \forall k} \pi(x)$$

Closed networks (continued)

- Throughput of node i

$$\gamma_i = \frac{G_{M_1,\ldots,M_k-1,\ldots,M_K}(\rho)}{\frac{\partial}{\partial \rho_i} G_{M_1,\ldots,M_K}(\rho)}$$

- Generating function

$$\begin{aligned}
\mathcal{G}(z_1,\ldots,z_K;\rho) &= \sum_{M_1,\ldots,M_K} G_{M_1,\ldots,M_K}(\rho) z_1^{M_1} \cdots z_K^{M_K} \\
&= G(\tilde{\rho}') \prod_{k=1}^{K} H_k(\tilde{\varrho}_k), \qquad \tilde{\rho}_i = \rho_i \prod_{k:i \in c_k} z_k
\end{aligned}$$

Closed networks (continued)

Example: One PS-link, one thinking stage

- The capacity of the link is denoted by C.
- The thinking stage is described as an infinite system of c-capacity servers.
- The generating function is

$$\mathcal{G}(z;\rho,\varrho) = \frac{e^{z\varrho/c}}{1 - z\rho/C}$$

Closed networks (example continued)

- Developing the generating function in Taylor series in z we can identify the $G_M(\rho, \varrho)$ and calculate the mean throughput on the link:

$$\gamma_M = \begin{cases} C, & M = 1 \\ \frac{c+C}{2c+C}\, C, & M = 2 \\ \frac{2c^2+2cC+C^2}{6c^2+4cC+C^2}\, C, & M = 3 \\ \frac{6c^3+6c^2C+3cC^2+C^3}{24c^3+18c^2C+6cC^2+C^3}\, C, & M = 4 \\ \frac{24c^4+24c^3C+12c^2C^2+4cC^3+C^4}{120c^4+96c^3C+36c^2C^2+8cC^3+C^4}\, C, & M = 5 \end{cases}$$

- For $c/C \to 0$ we have $\gamma_M \to C$. For $c/C \to \infty$ we have $\gamma_M \to C/M$.

State-dependent arrival rates and routing

- Extension to Processor Sharing networks with state dependent capacities, routing and arrival rates $\phi_i(x),\ p_{ij}(x),\ \nu_i(x)$.

- Assuming that for any state x the following traffic equations have a unique solution for the $\lambda_i(x)$,

$$\lambda_i(x) = \nu_i(x) + \sum_j p_{ij}(T^j x)\lambda_j(x) \qquad \text{(where } T^j x \equiv x + e_j\text{)}$$

- Define the traffic intensity at node i in state x: $\rho_i(x) = \lambda_i(x)/\mu_i$

State-dependent arrival rates and routing (continued)

- Denote $\psi_i(x) = \dfrac{\rho_i(T_i x)}{\phi_i(x)}$ (where $T_i x \equiv x - e_i$)

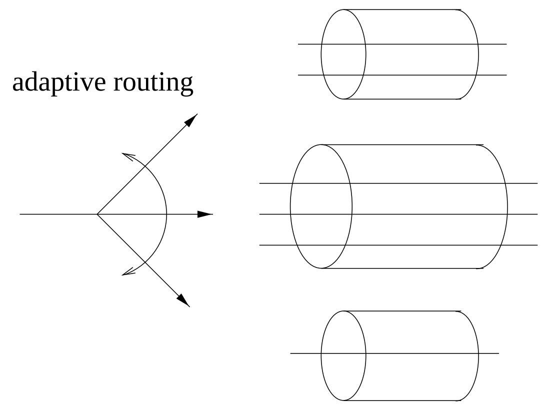
- Consider a PS network with state-dependent routing and arrival rates. The corresponding invariant measures are insensitive to the service time distributions at each node if and only if the $\psi_i(x)$ are balanced. In this case, an invariant measure is

$$\pi(x) = \psi_{i_1}(x)\psi_{i_2}(T_{i_1}x)\cdots\psi_{i_n}(T_{i_n}\cdots T_{i_1}x)$$

- If the allocations $\phi_i(x)$ are balanced then the $\psi_i(x)$ are balanced iff the $\lambda_i(x)$ are balanced (i.e., $\forall x : \lambda_i(x) = u(T^i x)/u(x)$ for some $u(x)$).

Parallel links with adaptive routing

- L parallel links
- Link i can serve at most N_i flows
- Adaptive stochastic routing $\nu_i(x)$
- A flow is rejected only if all links are saturated (i.e., for all other states x: $\sum_i \nu_i(x) = \lambda$).
- Insensitivity iff $$\nu_i(x) = \lambda \frac{N_i - x_i}{\sum_{l=1}^{L} (N_l - x_l)}$$

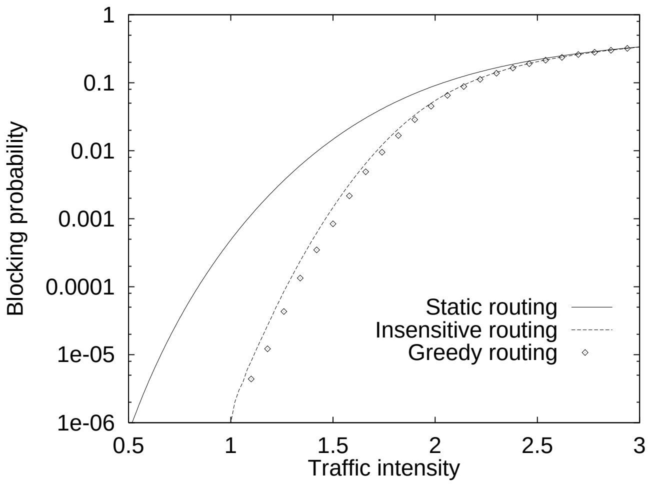


Adaptive routing (continued . . .)

- Stationary distribution and blocking probability

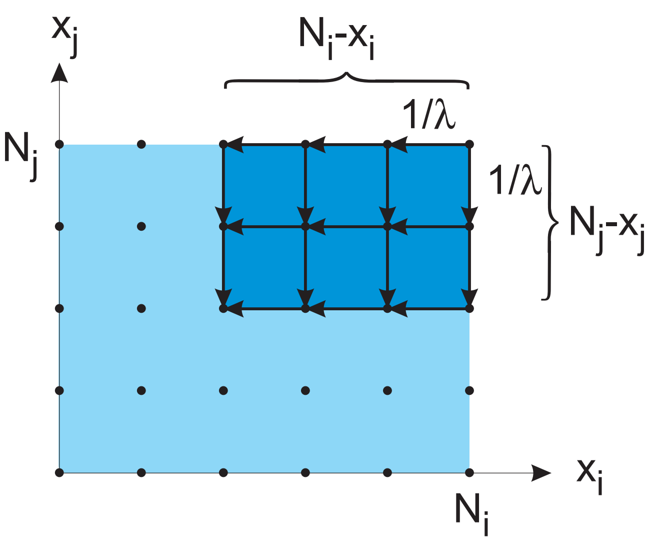
$$\pi(x) = \binom{\sum_{l=1}^{L}(N_l - x_l)}{N_1 - x_1, \ldots, N_L - x_L} \prod_{i=1}^{L} \left(\frac{\rho}{C_i}\right)^{x_i}$$

$$P = \frac{\pi(N_1, \ldots, N_L)}{\sum_{x_1=0}^{N_1} \cdots \sum_{x_L=0}^{N_L} \pi(x_1, \ldots, x_L)}$$



Adaptive routing (continued . . .)

- Where did the routing formula come from?
- The “shared resource” is the available traffic to be routed on different links
  - “mirror image” of the link sharing



$$u(x) = \binom{\sum_{l=1}^{L}(N_l - x_l)}{N_1 - x_1, \ldots, N_L - x_L} \frac{1}{\lambda^{\sum_l (N_l - x_l)}}$$

No efficient adaptive insensitive non-blocking routing

- Using Hausdorff's theorem M. Jonckheere showed that without any limit on the number of flows on links the general solution of the recursion is

$$u(x_1, x_2) = \int_z z^{x_1}(1-z)^{x_2} d\chi(z)$$

  where $\chi(z)$ is an arbitrary measure on $[0, 1]$.

- For instance if $d\chi(z)/dz = \delta(z-p)$ then $u(x_1, x_2) = p^{x_1}(1-p)^{x_2}$, i.e., static routing with $\nu_1 = p\lambda$, $\nu_2 = (1-p)\lambda$.

- With the above representation Jonckheere proved that there is no solution such that $\nu_1(x_1, 0) = u(x_1 + 1, 0)/u(x_1, 0) \to 0$ when $x_1 \to \infty$.

Open problems – future research

- Can balanced fairness be realized by a distributed algorithm?
- How close is balanced fairness to “classical” allocations?
- How sensitive are allocations that are not insensitive?
- New network topologies
- Background traffic in balanced routing
- Improved bounds