Minimum Transmission Energy Trajectories for a Linear Pursuit Problem

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Abstract. In this paper we study a pursuit problem in the context of a wireless sensor network, where the pursuer (i.e., mobile sink) trying to capture a pursuee (i.e., tracked object), moving with constant velocity, is always directly communicating with a sensor node in the very near proximity of the pursuee. Assuming that the sensor nodes can adjust their transmission power depending on the distance ρ between the pursuer and pursuee according to the usual power law $\rho^{-\alpha}$, the task is to find the optimal trajectory of the pursuer minimizing the total transmission energy. We approach this classical control theoretic problem by the method of dynamic programming. The cost function, describing the transmission cost with an optimal policy, factorizes into radial and angular functions. The partial differential equation governing the cost function can then be reduced to an ordinary differential equation for the angular function. This equation as well as the related optimal trajectories can be solved numerically. The qualitative behavior of the trajectories is also discussed. The trajectories are self-similar in the sense that any magnification of an optimal trajectory is also an optimal trajectory for different initial conditions.

1 Introduction

Pursuits are common in many areas, including predators that hunt for their preys, missiles that are heading towards their moving target, or a robot that is trying to reach (or at least get as close as possible to) its target to be monitored, etc. (see [1, 2] and references therein). Technically speaking, in a pursuit one particle travels along a specified curve, while a second pursues it, with a motion directed towards the first. When the pursuer travels faster than the pursued, the question then becomes: "At what point do the two meet?" "What is the capture point?" Besides, an interesting study can be made if there is a cost associated with the pursuit, and the task is to reduce this cost as much as possible. For example, a trivial objective can be to catch the target as soon as possible. The answer is typically given by defining the optimal strategy to drive the pursuer, or equivalently, defining the optimal curve of pursuit that should be followed. The curve of pursuit is simply the trajectory traced by the pursuer.

A typical assumption is that the pursuer is always heading right towards the target, i.e., with the paths of the pursuer and pursuee parameterized in time, the pursuee is always on the pursuer's tangent. This is because the pursuee's trajectory is not known in advance, either because the pursuer is not able to predict it, or the pursuee is actively trying to avoid the pursuer by changing its direction and speed adaptively. However, we concentrate on the problem where the pursuee is moving constantly along a straight line irrespective to the pursuer's behavior, also called as *linear* pursuit. (An excellent overview of the history of pursuit curves is found in a series of articles written by Arthur Bernhart, among which the first discusses pursuit curves where the pursued moves along a straight line [3].) Moreover, in our model the cost rate is related with the actual distance of the pursuer from the pursuee during the chase. The total energy of the *entire* pursuit is to be minimized, with the consumed power at each step being proportional to a given power of the relative distance between the pursuer and pursuee.

We adopt this pursuit problem to a wireless sensor networking scenario. As an example, consider a sensor networking application where sensor nodes detect any moving object within their sensing range, and report on it to a sink node. (For a general description of wireless sensor networks, please refer to [4–6].) Here we assume a single-hop network where all sensors send radio packets directly to the sink. The most important source of energy leakage in this scenario is the energy needed for radio communication. We assume that the radio transmission power obeys the well-known power-law $\rho^{-\alpha}$ as a function of the distance ρ, and the nodes are able to adjust their transmission power as needed. Since alerted nodes report periodically, the consumed energy at each time is related to the distance between the sink and the moving object. A pursuit problem can be defined if we allow the sink node to move freely. The pursuer in this case is the mobile sink, while the tracked object takes the role of the pursuee. Since the mobile sink is constantly communicating with the sensor nodes that are sensing the object (or, less likely, directly with that object), in order to reduce energy consumption, the task is to find the optimal pursuit curve that leads to minimal energy and thus extended network lifetime. (For a detailed description on the energy consumption, sink mobility and network lifetime in wireless sensor networks, please refer to [7].)

The task leads to a classical (non-stochastic) control theoretic problem. Differential equations for a linear pursuit are sometimes applied, where the pursued starts at rest and then moves along a straight line. In the simplest case, where the pursuer is always heading directly towards the pursuee, the equation of motion for the pursuer is then solvable by first setting the first derivative equal to a particular point. However, in our case the cost function is nonlinear. We approach the problem with the dynamic programming approach of Bellman [8], [9]. The state of the system is defined by the relative position of the pursuer and the pursuee. Associated with the state there is the cost function which represents the minimal cost from that state to the end when optimal curve of pursuit is followed. An analogous, but for telecommunications people more familiar concept is 'distance vector' which represents the length of the shortest path from a given node to the destination. In this networking context, dynamic programming principle is well-known from the solution of the shortest-path problem using the Bellman-Ford algorithm [10]. In our pursuit setting, we derive a partial differential equation for the cost function. Assuming that the transmission power obeys the power-law $\rho^{-\alpha}$, the partial differential equation reduces to an ordinary differential equation that can be solved numerically. When the cost function is known, the optimal trajectories can easily be calculated. In particular, when the problem is to catch the pursuee in minimum time (i.e., $\alpha = 0$), one easily infers that the optimal policy is to head with full speed towards where the pursuee is going (the meeting point) along a straight line. When large distances are very costly in terms of transmission power (i.e., α grows), the nature of the trajectory changes. For very large α the optimal trajectory at any instant heads towards the pursuee's current position in order to decrease the distance as quickly as possible.

Another consequence of the power-law dependent transmission power is that the optimal pursuit curves are self-similar: given an optimal trajectory from a given initial point, magnifying the trajectory, i.e., multiplying the distance from the origin (pursuee) of each point of the trajectory by a constant yields the optimal pursuit curve starting from the point where the original initial point is sent by the magnification transformation.

The rest of the paper is organized as follows. In Section 2 the investigated linear pursuit problem is formulated, and a (non-linear) partial differential equation is derived for the optimal trajectory, using the dynamic programming method. The way how to solve this equation is also shown. Section 3 presents numerical results for different initial parameter settings. Finally, Section 4 concludes the paper.

2 Pursuit, cost, optimal trajectory

2.1 Notation and problem formulation

Consider a linear pursuit game. Assume that the pursuee moves with constant speed v, and the maximal velocity of the pursuer is u. The pursuer is faster, thus the ratio $\nu = v/u$ is smaller than one. The positions of the pursuee and pursuer at time t are denoted by $\mathbf{s}(t)$ and $\mathbf{r}(t)$, respectively (see Fig. 1). In particular, we will assume that the pursuee moves along the x-axis at a constant velocity, i.e., $\mathbf{s}(t) = \mathbf{s}(0) + v\,t\,\mathbf{e}_1$. The *relative* position ($\boldsymbol{\rho}$) of the pursuer to the pursuee can be expressed as

$$\boldsymbol{\rho} = \mathbf{r} - \mathbf{s} = x\,\mathbf{e}_1 + y\,\mathbf{e}_2,$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the perpendicular unit vectors.

Assuming that the communication power (P) depends on the relative distance as

$$P = |\boldsymbol{\rho}|^{\alpha}, \quad \alpha \geq 0, \tag{1}$$

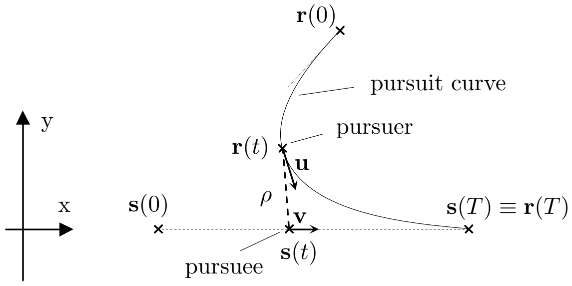


Fig. 1. Notations, pursuit curve.

the task is to navigate the pursuer, i.e., specify the trajectory $\mathbf{r}(t)$ up to some time T when the pursuer catches the pursuee, so that the total energy consumed is minimized. Thus, the *minimal* energy consumption during the 'chase' is given by

$$\varepsilon(\boldsymbol{\rho}) = \min_{|\mathbf{u}(t)| \le u, \forall t} \int_0^T |\mathbf{r}(t) - \mathbf{s}(t)|^\alpha \, dt, \tag{2}$$

where $\mathbf{u}(t) = \frac{d}{dt}\mathbf{r}(t)$, $\boldsymbol{\rho} = \mathbf{r}(0) - \mathbf{s}(0)$ and $\mathbf{r}(T) = \mathbf{s}(T)$. Our task is to find $\mathbf{u}(t)$ that realizes the minimum of (2).

Recalling the wireless sensor networking application mentioned earlier, the pursuit can be interpreted as follows. Assume that a mobile object is moving across the sensor field with a constant speed (see Fig. 2). Sensor nodes within

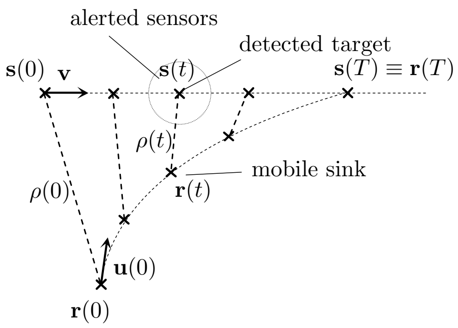


Fig. 2. Minimal energy 'chase' trajectory.

sensing range that detect the object in its very near proximity send packets to the sink node via their radio interface. We assume that the nodes are aware of their actual distance from the sink, and are able to adjust their radio transmission power according to (1) to reduce energy consumption. (For example, since we do not have any restriction on the sink node, we can assume that it is capable of broadcasting its position periodically to every node. Another solution would be that—instead of receiving the coordinates from the pursuer—the received signal strength could be used at the node to estimate the pursuer's distance from the node.) Assuming a mobile sink, the task is to find an optimal trajectory for the sink to minimize the *overall* energy consumption (i.e., (2)) of the network.

2.2 Catch in minimum time

In the case $\alpha = 0$ the transmission power is constant and the objective reduces to catching the pursuee in minimum time. It is easy to see that then optimal strategy for the pursuer is to go with maximal speed along a straight line to the point where it reaches the pursuee. If the pursuee at time $t = 0$ is at the origin, then at time t it is at point $(vt, 0)$. This point is reached by a pursuer at time t from all the points that lie on a circle with center $(vt, 0)$ and radius ut, see Fig. 3. The cost function $\varepsilon(\boldsymbol{\rho})$ at point $\boldsymbol{\rho} = (x, y)$ is then the time t that solves the equation

$$\sqrt{(x - vt)^2 + y^2} = ut.$$

The solution is

$$u\,\varepsilon(\boldsymbol{\rho}) = \frac{\sqrt{x^2 + (1 - \nu^2)\,y^2} - \nu x}{1 - \nu^2} = \rho\,\frac{\sqrt{1 - \nu^2 \sin^2\theta} - \nu\cos\theta}{1 - \nu^2}, \tag{3}$$

where in the latter form we have used polar coordinates, where $\rho = |\boldsymbol{\rho}|$ and θ is the angle between $\boldsymbol{\rho}$ and $\mathbf{e}_1$. Note that the expression factorizes into radial and angular factors. Rescaling ρ with a constant factor multiplies $\varepsilon(\boldsymbol{\rho})$ by the same factor. It follows that a magnification of an equivalue contour yields another equivalue contour as depicted in Fig. 3.

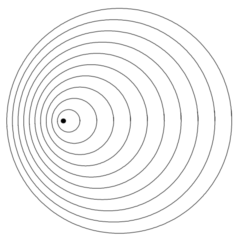


Fig. 3. Equivalue contours of $\varepsilon(\boldsymbol{\rho})$ are circles that are obtained by a magnification or contraction operation from each other.

2.3 Dynamic programming

For a general α we approach the problem with the method of dynamic programming. Let the pursuer choose the velocity $\mathbf{u}$ at time zero and proceed with this velocity over time interval dt. In order for the initial velocity to be optimal, we must have

$$\begin{aligned}\varepsilon(\boldsymbol{\rho}) &= \rho^\alpha\,dt + \min_{|\mathbf{u}|\le u} \varepsilon(\boldsymbol{\rho} + (\mathbf{u}-\mathbf{v})\,dt)\\ &= \rho^\alpha\,dt + \min_{|\mathbf{u}|\le u} \{\varepsilon(\boldsymbol{\rho}) + (\mathbf{u}-\mathbf{v})\cdot\nabla\varepsilon(\boldsymbol{\rho})\,dt\}\\ &= \rho^\alpha\,dt + \varepsilon(\boldsymbol{\rho}) - \mathbf{v}\cdot\nabla\varepsilon(\boldsymbol{\rho})\,dt + \min_{|\mathbf{u}|\le u} \mathbf{u}\cdot\nabla\varepsilon(\boldsymbol{\rho})\,dt.\end{aligned} \tag{4}$$

The minimum of the last term with respect to $\mathbf{u}$ is obtained when the direction of $\mathbf{u}$ is opposite to $\nabla\varepsilon(\boldsymbol{\rho})$ and $|\mathbf{u}| = u$. The minimum value attained is $-u|\nabla\varepsilon(\boldsymbol{\rho})|\,dt$. This leads to the first order (non-linear) partial differential equation for the function $\varepsilon(\boldsymbol{\rho})$,

$$\rho^\alpha - \mathbf{v}\cdot\nabla\varepsilon(\boldsymbol{\rho}) - u|\nabla\varepsilon(\boldsymbol{\rho})| = 0.$$

In component form this reads

$$\rho^\alpha - v\frac{\partial\varepsilon}{\partial x} - u\sqrt{\left(\frac{\partial\varepsilon}{\partial x}\right)^2 + \left(\frac{\partial\varepsilon}{\partial y}\right)^2} = 0.$$

Next we focus on how the solution of this equation can be obtained.

2.4 Solving the equation

Since the important parameter is the ratio (ν) of the speeds of the pursuee and pursuer, and not the absolute speeds, without loss of generality we may take $u = 1$. Then, with the notation $v = \nu u$, the equation reads

$$\rho^\alpha - \nu\frac{\partial\varepsilon}{\partial x} - \sqrt{\left(\frac{\partial\varepsilon}{\partial x}\right)^2 + \left(\frac{\partial\varepsilon}{\partial y}\right)^2} = 0. \tag{5}$$

This is most easily solved using polar coordinates introduced above, i.e. we solve $\varepsilon = \varepsilon(\rho, \theta)$. A solution is obtained with the separable trial (cf. the form of (3))

$$\varepsilon(\rho,\theta) = \tfrac{1}{1+\alpha}\,\rho^{\alpha+1}\varphi(\theta), \tag{6}$$

where the constant factor $\frac{1}{1+\alpha}$ is introduced for later convenience, and $\varphi(\theta)$ is an angular function yet to be found. With this trial we have

$$\begin{cases} \dfrac{\partial\varepsilon}{\partial x} = \rho^\alpha\big(\cos\theta\,\varphi(\theta) - \sin\theta\,\varphi'(\theta)/(1+\alpha)\big),\\[2ex] \dfrac{\partial\varepsilon}{\partial y} = \rho^\alpha\big(\sin\theta\,\varphi(\theta) + \cos\theta\,\varphi'(\theta)/(1+\alpha)\big).\end{cases}$$

Upon substitution in (5) the factor ρ^α is canceled and we are left with an ordinary differential equation for the angular function $\varphi(\theta)$,

$$\nu\big(\cos\theta\,\varphi(\theta) - \sin\theta\,\varphi'(\theta)/(1+\alpha)\big) + \sqrt{\varphi(\theta)^2 + \varphi'(\theta)^2/(1+\alpha)^2} = 1. \tag{7}$$

More explicitly, solved for $\varphi'(\theta)$ the differential equation reads[3]

$$\varphi'(\theta) = (1+\alpha)\,\frac{\nu \sin\theta - \sqrt{1 - 2\nu\varphi(\theta)\cos\theta - (1-\nu^2)\varphi^2(\theta)} - \nu^2\varphi(\theta)\sin\theta\cos\theta}{1-\nu^2\sin^2\theta}. \tag{8}$$

Because of symmetry, we have $\varphi'(0) = \varphi'(\pi) = 0$. The corresponding values $\varphi(0)$ and $\varphi(\pi)$ are readily solved from (7)[4],

$$\varphi(0) = \frac{1}{1+\nu}, \qquad \varphi(\pi) = \frac{1}{1-\nu}. \tag{9}$$

It is straightforward to check that the angular function, i.e. the coefficient of ρ, in (3) satisfies (8) for $\alpha = 0$, while an analytic solution for general α is not known. Equation (8) can, however, easily be solved numerically[5]. In Fig. 4 a family of solutions for $\varphi(\theta)$, corresponding to different values of α, $\alpha = 0,\ 1,\ 2,\ 5,$ and 25, are depicted for a fixed value of $\nu = \frac{1}{2}$ (with this value of ν we have $\varphi(0) = \frac{2}{3}$ and $\varphi(\pi) = 2$).

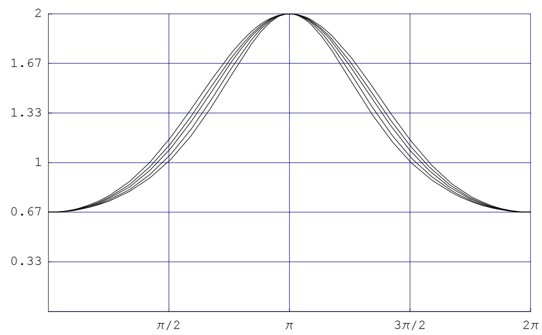


Fig. 4. Angular function $\varphi(\theta)$ for $\alpha = 1,\ 2,\ 5,\ 25$ (from top to bottom) with $\nu = \frac{1}{2}$.

3 Numerical results

Recalling that the velocity vector $\mathbf{u}$ of the pursuer is opposite to the direction of $\nabla\varepsilon(\boldsymbol{\rho})$ and that the pursuer always uses the full speed $|\mathbf{u}| = 1$, the trajectory

[3] One has to choose the minus sign for the square root; plus sign would lead to an imaginary solution.

[4] These results can be derived also as follows. When $\theta = 0$ or π, the optimal strategy is obviously to go straight along the x-axis at full speed towards the pursuee. Then, $\varepsilon(x\,\mathbf{e}_1) = \int_0^{|x|/(1\pm\nu)} (|x| - (1\pm\nu)\,t)^\alpha\,dt = \frac{1}{1\pm\nu}\int_0^{|x|}(|x|-y)^\alpha\,dy$, where $\pm$ stands for $\operatorname{sign} x$. The integration yields $\varepsilon(x\,\mathbf{e}_1) = \frac{1}{1\pm\nu}\frac{1}{1+\alpha}|x|^{1+\alpha}$ from which, in view of (6), result (9) follows.

[5] To guarantee numerical stability, the equation has to be solved backwards from π to 0; values in the range $\theta \in (\pi, 2\pi)$ are obtained by symmetry from those in range $\theta \in (0, \pi)$.

of the pursuer can be solved when the function $\varepsilon(\boldsymbol{\rho})$ is known. In the sequel, we denote the unit vector in the direction of $-\nabla\varepsilon(\boldsymbol{\rho})$ by $\boldsymbol{\epsilon}(\boldsymbol{\rho})$.

It is useful to note some general properties of the trajectories. First, from the separable form (6) it follows that the direction of $\nabla\varepsilon$ is a function of the angle θ only. It then follows that if $f(x, y) = 0$ defines the path of a trajectory, then also $f(cx, cy) = 0$ is a path for all $c > 0$. In other words, an arbitrary magnification or contraction of an optimal path results in another optimal path.

The trajectories can be solved either in moving coordinates (moving with the pursuee) or in fixed coordinates, i.e. one can solve either $\boldsymbol{\rho}(t)$ or $\mathbf{r}(t)$. These are determined by the differential equations

$$\frac{d}{dt}\boldsymbol{\rho}(t) = \boldsymbol{\epsilon}(\boldsymbol{\rho}(t)) - \mathbf{v},$$

$$\frac{d}{dt}\mathbf{r}(t) = \boldsymbol{\epsilon}(\mathbf{r}(t) - \mathbf{v}\, t).$$

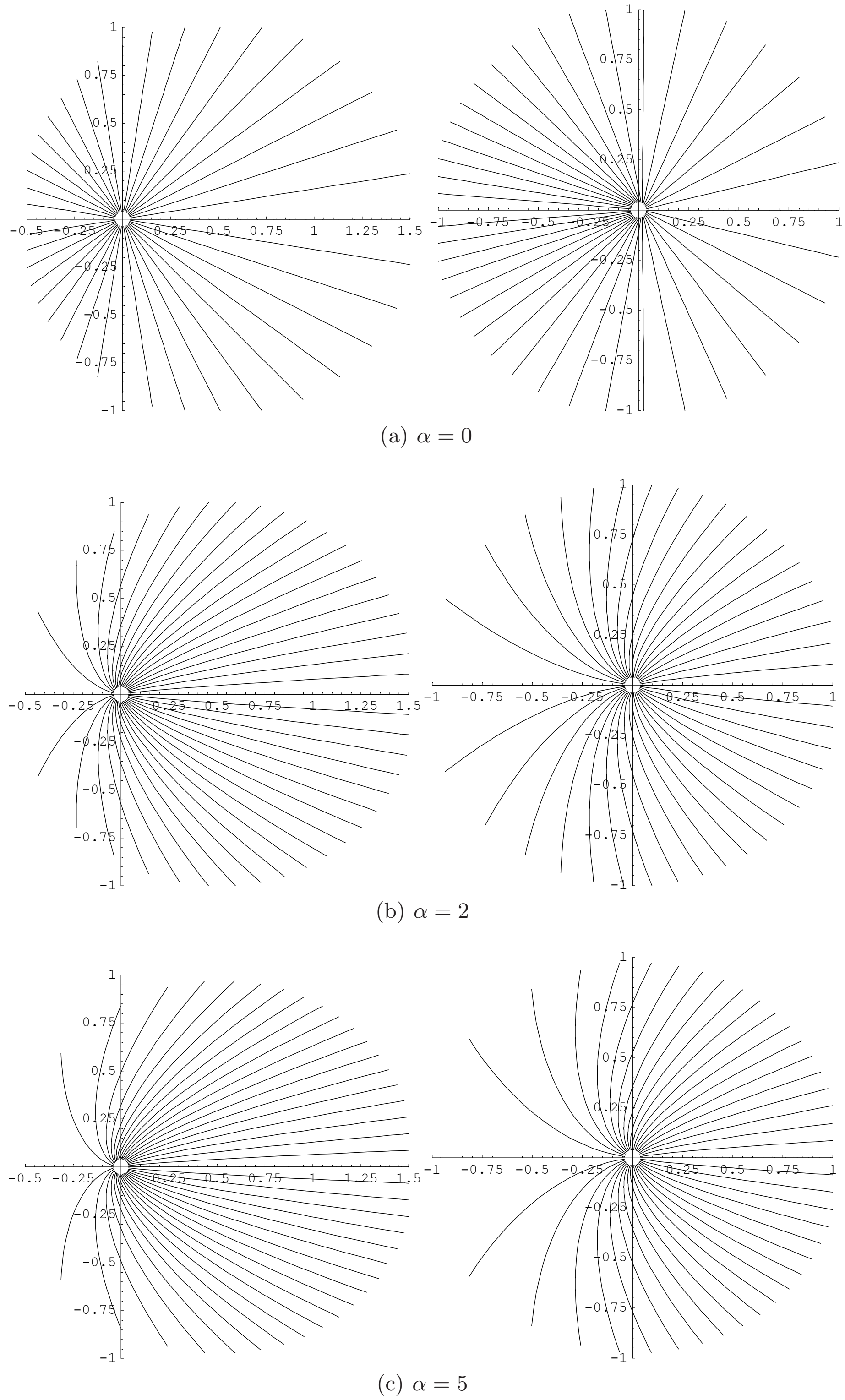
In Fig. 5 we give examples of the trajectories for three different values of α, $\alpha = 0, 2, 5$, with $\nu = \frac{1}{2}$. The trajectories are drawn for $t \in (0, 1)$ for a pursuer that reaches the pursuee at time $t = 1$. For $\alpha = 0$ the trajectories are straight lines as they should.

Looking at the trajectories in fixed coordinates (the right hand graphs), one notes an intuitively obvious behavior. Regarding that at time $t = 0$ the pursuee is at point $(-\frac{1}{2}, 0)$, we find that in the case $\alpha = 0$ the pursuer does not head to 'where the pursuee is' but directly to 'where the pursuee is going'. When the value of α increases the optimal trajectory more and more turns to the one that heads to 'where the pursuee currently is'. This happens in order to decrease the distance between the pursuer and the pursuee as quickly as possible; this is advantageous because for a large α the objective function decreases very rapidly as the distance decreases.

4 Conclusions

We studied a linear pursuit problem with an application example of target detection and tracking in a wireless sensor networking scenario using a mobile sink. We identified the optimal trajectory that should be followed by the sink to minimize the energy consumption in the network. The energy of radio transmission to be minimized is defined by a cost rate that obeys the well-known power-law $\rho^{-\alpha}$ as a function of the distance ρ between the mobile sink (pursuer) and the moving target (pursuee).

We approached the problem by the method of dynamic programming. We showed that the cost function, describing the radio transmission cost with an optimal policy, factorizes into radial and angular functions. The partial differential equation governing the cost function reduces to an ordinary differential equation for the angular function. This equation as well as the related optimal trajectories can be solved numerically.



(a) $\alpha = 0$

(b) $\alpha = 2$

(c) $\alpha = 5$

Fig. 5. Trajectories in moving and fixed coordinates.

Parameter α gives a great flexibility to this model. When α is set to zero, the problem reduces to the task of catching the target as soon as possible. The resulting optimal trajectories in this case are straight lines leading directly towards the rendezvous-point. On the other hand, when α is set to two or more, the cost function is a realistic model for the energy requirement of radio transmission. The resulting optimal trajectory ensures in this case the minimum overall energy consumption in the network. When α is large, the optimal pursuit is the one where the sink is always heading right towards the target's actual position, trying to reduce the relative distance as much as possible.

An interesting consequence is that, having the power-law dependent cost function, the optimal pursuit curves are self-similar in the sense, that any magnification of the curve results in an optimal trajectory as well, but for different initial conditions.

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