Calculation of time-dependent blocking probabilities

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Abstract

We consider the calculation of time-dependent blocking probabilities in the M/M/n/n loss system. We present a new method which allows one to find the temporal evolution of the probability of any given state, and, in particular, that of the blocking state without the need to solve the probabilities of all the other states or to find the eigenvalues and eigenvectors of a very large matrix corresponding to the full system. The problem is motivated by a dynamic VP bandwidth management scheme of an ATM network with full traffic segregation.

1. INTRODUCTION

In the dynamic VP bandwidth management scheme developed by Mocci et al. [1]-[4] the bandwidth allocation for a VP in an ATM network is adjusted at regular time intervals. At the beginning of an interval the system occupancy is observed and new VP capacity allocation is done in such a way that the expected time average of the blocking probability in the interval will be less than a predefined limit. In the simplest case of full traffic segregation, each VP carries only one type of traffic and the problem within a VP is one-dimensional.

Thus one is led to consider the following problem. New calls arrive according to a Poisson process with rate λ to a loss system with n trunks. Exponential holding time with mean $1/\mu$ is assumed. Given the number of calls, i, in progress at time 0 the task is to calculate the probability $P_{n|i}(t)$ of finding the system in state n at time t. The average blocking probability in an interval of length T is then $(1/T)\int_0^T P_{n|i}(t)dt$.

The time-dependent state probabilities are determined by the system of differential equations

$$\begin{aligned}
\dot{P}_n(t) &= \lambda P_{n-1}(t) - n\mu P_n(t), \\
\dot{P}_k(t) &= \lambda P_{k-1}(t) - (\lambda + k\mu)P_k(t) + \\
&\quad (k+1)\mu P_{k+1}(t), \quad k = 1, \ldots, n-1, \\
\dot{P}_0(t) &= \mu P_1(t) - \lambda P_0(t),
\end{aligned} \tag{1}$$

with the initial conditions $P_i(0) = 1$ and $P_k(0) = 0$ for $k \neq i$. These equations can, of course, be solved numerically. However, in order to find $P_n(t)$, the equations have to

be solved for all values, $k = 0, \ldots, n$. Equivalently, one can consider the above system as a vector equation $\dot{\mathbf{P}}(t) = \mathbf{P}(t)\mathbf{Q}$, with $\mathbf{P}(t) = (P_0(t), P_1(t), \ldots)$ and the coefficient matrix $\mathbf{Q}$ being the generator matrix of the Markov process, and solve it with the aid of the eigenvectors of $\mathbf{Q}$. Both of these methods become impracticable as n becomes very large. Perhaps the best known method for the solution is to start from the formal solution $\mathbf{P}(t) = \mathbf{P}(0)\exp(\mathbf{Q}t)$ and apply the so called uniformization method for the calculation of the matrix exponent function, see [5].

Our aim is to develop a method which allows the calculation $P_{n|i}(t)$ separately without the need to consider the evolution of all the other states. The computational effort of our method is only weakly dependent on the size of the system, thus enabling us to calculate the time-dependent blocking probabilities even for very large systems.

2. INFINITE SYSTEM

We consider first an infinite system, where the state probabilities can be deduced easily. The number of calls $N(t|i)$ in progress at time t, given that there are i calls in progress at $t = 0$, is the sum of two independent random variables: the number of calls $N_0(t|i)$ surviving from the initial set of i calls and the number $N_1(t)$ of calls arrived in $(0, t)$ and still in progress at time t. For each call from the initial set the survival probability at t is

$$p_t = e^{-\mu t}.$$

Thus $N_0(t|i)$ is binomially distributed, $N_0(t|i) \sim \mathrm{Bin}(i, p_t)$. Similarly, taking into account the survival probabilities, we reason that $N_1(t)$ is the number of arrivals in $(0, t)$ from an inhomogeneous Poisson process with intensity $\lambda(t') = \lambda e^{-\mu(t-t')}$ with $t' \in (0, t)$. Thus $N_1(t)$ is Poisson distributed with mean $a(1 - p_t)$, $N_1(t) \sim \mathrm{Poisson}(a(1 - p_t))$, where $a = \lambda/\mu$ is the offered traffic intensity.

The time-dependent probability of any state k in the infinite system, $P^{\infty}_{k|i}(t)$, can be obtained in two different ways: by the numerical convolution of the binomial and Poisson distributions or by an approximate inversion (e.g. by the probability shift method) of the probability generating function of the distribution of $N(t|i)$,

$$N(z, t|i) = (1 - p_t + p_t z)^i e^{(z-1)(1-p_t)a}.$$

Let us briefly consider the approximate inversion with the probability shift method. The shifted mean and variance of the distribution are

$$\begin{aligned} m(z, t|i) &= a(1 - p_t)z + \frac{ip_t z}{1 - p_t + p_t z}, \\ v(z, t|i) &= a(1 - p_t)z + \frac{ip_t(1 - p_t)z}{(1 - p_t + p_t z)^2}. \end{aligned}$$

In order to estimate $P^{\infty}_{k|i}(t)$ we determine $z = z_{k|i}(t)$ such that $m(z, t|i) = k$. This leads to

$$z_{k|i}(t) = \frac{\sqrt{(a(1 - p_t)^2 + (i - k)p_t)^2 + 4aip_t(1 - p_t)^2} - (a(1 - p_t)^2 + (i - k)p_t)}{2ap_t(1 - p_t)}.$$

The probability shift argument finally gives (cf. [6], (5.4.5))

$$P^{\infty}_{k|i}(t) \approx \left. \frac{z^{-k} N(z,t|i)}{\sqrt{2\pi v(z,t|i)}} \right|_{z\,=\,z_{k|i}(t)} .$$

3. FINITE SYSTEM

The analysis of the finite system can be based on two observations. First, the equations (1) for a finite system are the same as those of an infinite system for the range $k = 0, \ldots, n$ except that in the equation for $k = n$ the terms representing transitions between states n and $n+1$ are missing. Second, the system of equations (1), as well as its counterpart for an infinite system, are linear. Our strategy then is to consider a modified infinite system, where an extra source term $s(t)$ is added to one of the equations, m, say, in such a way that the resulting equations for $k = 0, \ldots, n$ in the modified system are exactly the same as those for the finite system.

The effect of the source term on the solution of the infinite system can be easily deduced by the linearity of the equations. Addition of a source term $s(t)$ to equation m gives rise to an additional term

$$\int_0^t s(u) P^{\infty}_{k|m}(t-u)du$$

in the time-dependent state probabilities, where $P^{\infty}_{k|m}(t)$ is the known solution for the initial value problem in the infinite system. This term represents a superposition, with weight $s(u)$, of the probabilities implied by unit probability mass addition to state m at different times u.

In principle, we can carry out the strategy for any choice of the state m with $m \geq n$. Here we consider the two most reasonable choices, $m = n$ or $m = n+1$. In the first case, the external source injects probability mass to the state n at a rate $s(t)$ chosen exactly to compensate for the net probability flow from state n to state $n+1$. The total probability mass in the whole infinite system is not conserved but the mass in the states $(k = 0, \ldots, n)$ remains constant (equal to 1). The second approach has a similar idea. In this case we add an external source injecting probability mass to the state $n+1$ in such a way that the probability of state $n+1$ keeps a fixed relation to that of state n in order to guarantee zero net flow of probability between these states.

3.1. First approach

In the first approach, we modify equation n of the infinite system by adding a source term $s(t)$ as follows,

$$\begin{aligned} \dot{P}_n(t) &= \lambda P_{n-1}(t) - (\lambda + n\mu) P_n(t) + \\ &\quad (n+1)\mu P_{n+1}(t) + s(t). \end{aligned}$$

The state diagram with the added source term is depicted in Fig. 1. Now, by choosing $s(t)$ to be

$$s(t) = \lambda P_n(t) - (n+1)\mu P_{n+1}(t) \tag{2}$$

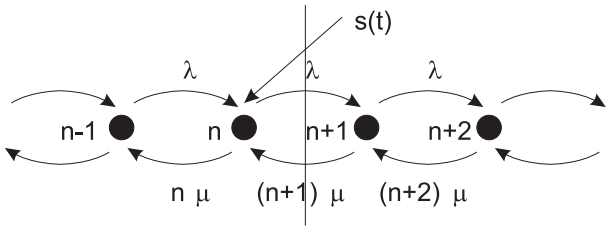

Figure 1. In the first method a source is added to state n such that the probability flow across the control surface is zero.

this equation reduces to that of the finite system. Referring to Fig. 1 this condition means that the probability flow across the control surface between states n and $n+1$ is zero. As noted before, the effect of the source term on the state probabilities can be accounted for by an extra term,

$$P_{k|i}(t) = P^{\infty}_{k|i}(t) + \int_0^t s(u) P^{\infty}_{k|n}(t-u)du. \tag{3}$$

Thus the condition (2) for $s(t)$ becomes

$$s(t) = R^{\infty}_{n|i}(t) + \int_0^t s(u) R^{\infty}_{n|n}(t-u)du, \tag{4}$$

where

$$R^{\infty}_{n|i}(t) = \lambda P^{\infty}_{n|i}(t) - (n+1)\mu P^{\infty}_{n+1|i}(t)$$

is the net leak rate from state n to state $n+1$ at time t in an infinite system starting from the initial state i at time 0. Solution of the integral equation (4) for $s(t)$ and substitution into (3) gives the desired time-dependent state probabilities.

3.2. Second approach

In the second approach, we add the source term $s(t)$ to the equation of state $n+1$ of an infinite system, see Fig. 2. Then the state probabilities can be written in an analogous fashion,

$$P_{k|i}(t) = P^{\infty}_{k|i}(t) + \int_0^t s(u) P^{\infty}_{k|n+1}(t-u)du, \tag{5}$$

the only difference being that now the impulse response function is $P^{\infty}_{k|n+1}(t-u)$. The condition for determining $s(t)$ in this case reads

$$\lambda P_{n|i}(t) = (n+1)\mu P_{n+1|i}(t), \tag{6}$$

i.e., the probability flows between states n and $n+1$ are required to cancel, whence the equation for state n again reduces to that of the finite system. By substituting (5) into (6) the latter can be rewritten as

$$R^{\infty}_{n|i}(t) = -\int_0^t s(u) R^{\infty}_{n|n+1}(t-u)du. \tag{7}$$

Again, solution of this integral equation for $s(t)$ and substitution into (5) gives the desired time-dependent state probabilities.

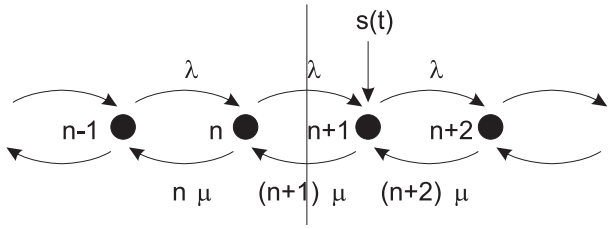

Figure 2. In the second method a source is added to state $n+1$. The probability flow across the control surface is again required to be zero.

4. NUMERICAL CONSIDERATIONS

4.1. Solution of the equations

In order to calculate the time-dependent probability of any given state, one has to solve the source term from an integral equation, (4) or (7), and then calculate another integral, (3) or (5), with this source term. In these integrals there appear kernel functions, which are known from the solution for an infinite system, and which can be calculated and represented in an appropriate numerical form, once and for all.

As the integral equations (4) or (7) for time t only depend on values of $s(u)$ for $u < t$, i.e., they are Volterra equations of the second and the first kind, respectively, we can solve the values of $s(t)$ sequentially. We outline a simple numerical scheme. To be more specific, we consider equation (7) as an example. Let us discretise the time

$$t_j = j\Delta t, \quad j = 0, 1, \ldots$$

and use a piecewise linear representation for $s(t)$

$$s(t) \approx \sum_{j>0} s_j w(t/\Delta t - j)$$

with $s_j = s(j\Delta t)$ and

$$w(t) = \begin{cases} 1 - |t|, & -1 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

The right hand side of (7) at time $t = k\Delta t$ can now be written as $\sum_{j=0}^{k} s_j A_{k-j}$, where

$$A_k = \begin{cases} -\Delta t \int_{-1}^{+1} w(u) R^{\infty}_{n|n+1}((k+u)\,\Delta t)\,du, & k > 0, \\ -\Delta t \int_{0}^{+1} w(u) R^{\infty}_{n|n+1}(u\,\Delta t)\,du, & k = 0. \end{cases}$$

Denoting further

$$b_k = R^{\infty}_{n|i}(k\Delta t),$$

equation (7) at time $k\Delta t$ becomes

$$b_k = \sum_{j=0}^{k} s_j A_{k-j},$$

from which we can solve

$$s_k = \frac{1}{A_0}\Big(b_k - \sum_{j=0}^{k-1} s_j A_{k-j}\Big), \qquad k = 0, 1, \ldots$$

In Fig. 3 an example of the application of the method is given. The evolution of the time-dependent blocking probability $P_{117|105}(t)$ is depicted for a system of size $n = 117$ starting from the state $i = 105$ with an offered traffic intensity of $a = 100$. For this system the stationary Erlang blocking probability is just below 1 %. Note the "overshooting" of the probability which is due to an initial state being well above the mean a. The average blocking probability can be obtained by averaging this instantaneous blocking probability over a given interval $(0, T)$.

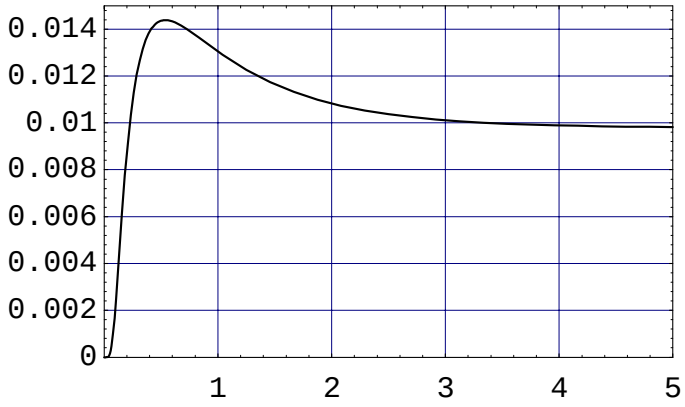

Figure 3. Blocking probability in a system of size $n = 117$ starting from the state $i = 105$. Offered traffic intensity $a = 100$. Time (horizontal axis) is given in units of $1/\mu$.

4.2. Asymptotic properties

It is of interest to note that though the calculation of the time-dependent blocking probability $P_{n|i}(t)$ requires the solution of the Volterra integral equations, its integral over t can be obtained from a simple consideration and only requires the solution of a system of ordinary linear equations. To see this, note first that, irrespective of the initial state i, asymptotically we have $P_{n|i}(t) \to B(n, a)$ as $t \to \infty$, where $B(n, a)$ is the Erlang loss function. Then the integral

$$v_i = \lambda \int_0^\infty (P_{n|i}(t) - B(n, a))dt \tag{8}$$

represents the expected number of additional lost calls due to starting the system from the specific state i rather than from the equilibrium, i.e. the 'cost' of state i. This is precisely the quantity that is determined by the Howard equations in the theory of Markov decision processes, see e.g. [7]. For the continuous time process, the equations can be written in the vector form

$$\mathbf{r} - r + \mathbf{v}\mathbf{Q}^{\mathrm{T}} = 0, \tag{9}$$

where $\mathbf{v}$ is the (row) vector of the state dependent costs, $\mathbf{Q}$ is the same generator matrix as before, $\mathbf{r}$ represents the state dependent loss rates, and r is the average loss rate. In our case, the nth component of $\mathbf{r}$ is λ and other components are zero. From (9) one can solve r and $\mathbf{v}$ up to an additive constant in the v_i. The constant is arbitrary but can be fixed by e.g. the condition $\boldsymbol{\pi}\mathbf{v}^{\mathrm{T}} = 0$, where $\boldsymbol{\pi}$ is the stationary distribution of the system, determined by $\boldsymbol{\pi}\mathbf{Q} = 0$, which $\mathbf{P}(t)$ asymptotically tends to. The v_i defined in (8) satisfy this condition since $\sum_i \pi_i P_{n|i}(t)$ represents the probability of the system being in state n at time t given it started from equilibrium at $t = 0$, and is identically equal to $B(n, a)$. The solution of the Howard equation (9) for our loss system is well known. First, we must have $r = \lambda B(n, a)$. Second, as shown by Krishnan [8], the differences between the state costs obey the simple formula $v_{i+1} - v_i = B(n, a)/B(i, a)$. It is then an easy task to find the values of the v_i such that $\boldsymbol{\pi}\mathbf{v}^{\mathrm{T}} = 0$.

As the value v_i of the integral (8) is known, we get an estimate for the average blocking probability valid for large T

$$\frac{1}{T}\int_0^T P_{n|i}(t)dt \approx B(n, a) + \frac{v_i}{\lambda T}.$$

5. DISCUSSION

We have presented a method which transforms the task of finding the time-dependent blocking probability from that of solving a set of n simultaneous differential equations to that of solving a *single* integral equation and evaluating a *single* integral with this solution. Apparently the method is independent of the size n of the system. In fact, however, there is a slight dependency on n since the kernel functions are more "peaky" for large n thus necessitating the use of denser grid for the numerical solution for a given accuracy. Despite of this dependency we find the new method computationally more efficient than the direct solution of the differential equations.

We have also compared the computational efficiency of the proposed method with the uniformization method referred to in the Introduction. Our studies indicate that for the calculation of the average blocking probability with a given accuracy, the uniformization method is more efficient for small systems but for larger systems the proposed method becomes more efficient. In our implementation the cross over occurred at a size of around $n = 200$ when the accuracy was 10^{-4}.

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