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Blocking probabilities in a transient system

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Abstract. *The dynamic VP bandwidth management scheme developed by Mocci et al. in a series of papers calls for the calculation of the average blocking probability over a finite time interval of a system in a transient state. We develop a "precision weapon" which allows one to find the temporal evolution of the probability of the blocking state without the need to solve the probabilities of all the other states or to find the eigenvalues and eigenvectors of a very large matrix corresponding to the full system.*

1 Introduction

In the dynamic VP bandwidth management scheme developed by Mocci et al. [1]-[4] the bandwidth allocation for a VP in an ATM network is adjusted at regular time intervals. At the beginning of an interval the system occupancy is observed and new VP capacity allocation is done in such a way that the expected time average of the blocking probability in the interval will be less than a predefined limit. In the simplest case of full traffic segregation, each VP carries only one type of traffic and the problem within a VP is one-dimensional.

Thus one is led to consider the following problem. New calls arrive according to a Poisson process with rate λ to a loss system with n trunks. Exponential holding time with mean $1/\mu$ is assumed. Given the number of calls i in progress at time 0 the task is to find the time dependent probability $P_{n|i}(t)$ of state n. The average blocking probability in an interval of length T is then $(1/T)\int_0^T P_{n|i}(t)dt$.

The time dependent state probabilities are determined by the system of equations

$$\begin{aligned}\dot{P}_n(t) &= \lambda P_{n-1}(t) - n\mu P_n(t) \\ \dot{P}_k(t) &= \lambda P_{k-1}(t) - (\lambda + k\mu)P_k(t) + (k+1)\mu P_{k+1}(t) \qquad k = 1,\ldots,n-1 \\ \dot{P}_0(t) &= \mu P_n(t) - \lambda P_0(t)\end{aligned} \tag{1}$$

with the initial conditions $P_i(0) = 1$ and $P_k(0) = 0$ for $k \neq i$. Equivalently, one can consider the above system as a vector equation $\dot{\mathbf{P}}(t) = \mathbf{A}\mathbf{P}(t)$ and solve it with the aid of the eigenvectors of $\mathbf{A}$. Both of these methods become impracticable as n becomes very large.

Our aim is to develop a method which allows the calculation $P_{n|i}(t)$ separately without the need to consider the evolution of all the other states. The computational effort of our method is essentially independent of the size of the system, thus enabling us to calculate the time dependent blocking probabilities even for very large systems. We start by first considering an infinite system.

2 Infinite system

In an infinite system ($n = \infty$) we can easily derive a good approximation for the time dependent state probabilities. The number of calls $N(t|i)$ in progress at time t is the sum of two independent random variables: the number of calls $N_0(t|i)$ surviving from the initial set of i calls and the number $N_1(t)$ of those calls arrived in $(0, t)$ that still survive at time t. For each call from the initial set the survival probability at t is

$$p_t = e^{-\mu t}. \tag{2}$$

Thus $N_0(t|i)$ is binomially distributed, $N_0(t|i) \sim \mathrm{Bin}(i, p_t)$. Similarly, taking into account the survival probabilities, we reason that $N_1(t)$ is the number of arrivals in $(0, t)$ from an inhomogeneous Poisson process with intensity $\lambda(t') = \lambda e^{-\mu(t-t')}$ with $t' \in (0, t)$. Thus $N_1(t)$ is Poisson distributed with mean $a(1 - p_t)$, $N_1(t) \sim \mathrm{Poisson}(a(1 - p_t))$, where $a = \lambda/\mu$ is the offered traffic intensity.

The time dependent probability of any state k in the infinite system, $P^{\infty}_{k|i}(t)$, can be obtained in two different ways: by the numerical convolution of the binomial and Poisson distributions (see Appendix) or by the approximate inversion of the generating function (probability shift method). With the above observations we can immediately write the generating function of the distribution of $N(t|i)$,

$$N(z, t|i) = (1 - p_t + p_t z)^i e^{(z-1)(1-p_t)a}. \tag{3}$$

The shifted mean and variance of the distribution are

$$\begin{aligned} m(z, t|i) &= a(1 - p_t)z + \frac{i p_t z}{1 - p_t + p_t z}, \\ v(z, t|i) &= a(1 - p_t)z + \frac{i p_t (1 - p_t) z}{(1 - p_t + p_t z)^2}. \end{aligned} \tag{4}$$

In order to estimate $P^{\infty}_{k|i}(t)$ we determine $z = z_{k|i}(t)$ such that $m(z, t|i) = k$. This leads to

$$z_{k|i}(t) = \frac{\sqrt{(a(1 - p_t)^2 + (i - k)p_t)^2 + 4aip_t(1 - p_t)^2} - (a(1 - p_t)^2 + (i - k)p_t)}{2ap_t(1 - p_t)}. \tag{5}$$

The probability shift argument finally gives (cf. [5], (5.4.5))

$$P^{\infty}_{k|i}(t) \approx \left. \frac{z^{-k} N(z,t|i)}{\sqrt{2\pi v(z,t|i)}} \right|_{z\,=\,z_{k|i}(t)} . \tag{6}$$

3 Finite system

The analysis of the finite system can be based on a few observations. First, the finite system ($k = 0, \ldots, n$) differs from the infinite one only in that the terms representing transitions between states n and $n+1$ in the first of equations (1) are missing. Second, the system of equations (1) is linear (both in the case of an infinite and a finite system). Thus we can try to construct a solution to the finite case by superimposing solutions for the infinite system, with appropriate initial conditions, in such a way that the superposed system satisfies equations (1). There are at least two ways to do this.

In the first we add an external source injecting probability mass to the state n at a rate $s(t)$ chosen to exactly compensate for the net probability flow from state n to state $n+1$. The total probability mass in the whole infinite system is not any more conserved but the mass in the states ($k = 0, \ldots, n$) remains constant (equal to 1).

The second has a similar idea. In this case we add an external source injecting probability mass to the state $n+1$ in such a way that the probability of state $n+1$ keeps a fixed relation to that of state n in order to guarantee zero net flow of probability between these states, i.e., the probability flow associated with upward transitions is reflected back by the corresponding downward transitions.

3.1 First approach

Let us analyse these alternative approaches in more detail. In both cases, what we actually solve are infinite systems. In the first approach we modify equation n of the infinite system by adding a source term as follows,

$$\dot{P}_n(t) = \lambda P_{n-1}(t) - (\lambda + n\mu) P_n(t) + (n+1)\mu P_{n+1}(t) + s(t) \tag{7}$$

Now, choosing $s(t)$ to be

$$s(t) = \lambda P_n(t) - (n+1)\mu P_{n+1}(t) \tag{8}$$

this equation reduces to that of the finite system. The only problem here is that the $P_k(t)$ represent the state probabilities of the modified system, which in turn depend on $s(t)$. But as the system is linear we can express these probabilities as a superposition of solutions for the infinite system with different initial conditions. Emphasizing again the dependence on the initial states and noting that $P^{\infty}_{k|n}(t-u)$ is the "impulse response" or "Green's function" for an addition of probability mass 1 at time u to the state n, we have

$$P_{k|i}(t) = P^{\infty}_{k|i}(t) + \int_0^t s(u) P^{\infty}_{k|n}(t-u)\, du. \tag{9}$$

Thus the condition (8) for $s(t)$ becomes

$$s(t) = R^{\infty}_{n|i}(t) + \int_0^t s(u) R^{\infty}_{n|n}(t-u)du, \tag{10}$$

where

$$R^{\infty}_{n|i}(t) = \lambda P^{\infty}_{n|i}(t) - (n+1)\mu P^{\infty}_{n+1|i}(t) \tag{11}$$

is the net leak rate from state n to state $n+1$ at time t in an infinite system starting from the initial state i at time 0. With the approximation (6) the leak rates are easily calculable functions, equation (10) can be solved numerically, and finally the evolution of state probabilities, e.g. that of the blocking state n, are obtained from (9).

3.2 Second approach

In the second approach we add the source term $s(t)$ to the equation of state $n+1$ of an infinite system. Then the state probabilities can be written in an analogous fashion,

$$P_{k|i}(t) = P^{\infty}_{k|i}(t) + \int_0^t s(u) P^{\infty}_{k|n+1}(t-u)du \tag{12}$$

the only difference being that now the impulse response function is $P^{\infty}_{k|n+1}(t-u)$. The condition for determining $s(t)$ in this case reads

$$\lambda P_{n|i}(t) = (n+1)\mu P_{n+1|i}(t), \tag{13}$$

i.e., the probability flows between states n and $n+1$ are required to cancel, whence the equation for state n again reduces to that of the finite system. By substituting (12) into (13) the latter can be rewritten as

$$R^{\infty}_{n|i}(t) = -\int_0^t s(u) R^{\infty}_{n|n+1}(t-u)du. \tag{14}$$

4 Numerical considerations

We outline a simple numerical scheme for the solution of equations of type (10) or (14). Formally they are integral equations, but the fact that the equations at time t only depend on values of $s(u)$ for $u < t$ enables us to solve the values of $s(t)$ sequentially. To be more specific, we consider equation (14) as an example. Let us discretize the time

$$t_j = j\Delta t, \quad j = 0, 1, \dots \tag{15}$$

and use a piecewise linear representation for $s(t)$

$$s(t) \approx \sum_{j>0} s_j w(t - j\Delta t) \tag{16}$$

with $s_j = s(j\Delta t)$ and

$$w(t) = \begin{cases} 1 - |t| & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

The right hand side of (14) at time $t = k\Delta t$ can now be written as $\sum_{j=0}^{k} s_j A_{k-j}$, where

$$A_k = \begin{cases} -\Delta t \displaystyle\int_{-1}^{+1} w(u) R^{\infty}_{n|n+1}((k+u)\,\Delta t)\,du, & k > 0, \\ \\ -\Delta t \displaystyle\int_{0}^{+1} w(u) R^{\infty}_{n|n+1}(u\,\Delta t)\,du, & k = 0. \end{cases} \tag{18}$$

Denoting further

$$b_k = R^{\infty}_{n|i}(k\Delta t) \tag{19}$$

equation (14) at time $k\Delta t$ becomes

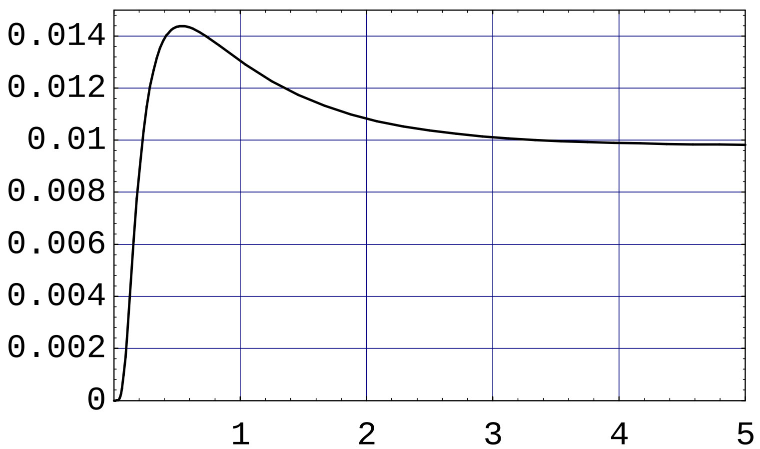
$$b_k = \sum_{j=0}^{k} s_j A_{k-j} \tag{20}$$

from which we can solve

$$s_k = \frac{1}{A_0}\Big(b_k - \sum_{j=0}^{k-1} s_j A_{k-j}\Big) \qquad k = 0, 1, \ldots \tag{21}$$

In the figure below the evolution of the blocking probability $P_{117|105}(t)$ is depicted for a system of size $n = 117$ starting from the state $i = 105$ with an offered traffic intensity of $a = 100$. For this system the stationary Erlang blocking probability is just below 1 %. Note the "overshooting" of the probability which is due to an initial state being well above the mean a.

The curve was obtained with a direct numerical solution of the set of 118 equations (1). The solution provided by the above simplified scheme, which directly calculates $P_{117|105}(t)$, was virtually indistinguishable when the time step $\mu\Delta t = 0.05$ was used.



References

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Appendix: convolution algorithm

Though approximation (6) is quite accurate, one may wish to calculate the probabilities and leak rates of an infinite system exactly by a numerical convolution of the binomial and Poisson distribution. Since the terms in both of these distributions can be calculated recursively, one can evaluate the convolution also recursively, in "one sweep", without having to save the whole distributions. For instance, in Mathematica language the required expression is

```
cnv[i_,a_,p_,k_] :=
 If[ p==0, ((1-p)a)^k/k! Exp[-(1-p)a],
     Module[ {trm=p^i*Exp[-(1-p)a],ii=i,j=0,sum},
              While[ii>k, ii--; trm*=(ii+1)/(i-ii)(1/p-1)];
              While[j<k-ii, j++; trm*=(1-p)a/j];
              sum=trm;
              While[ii-->0, sum+=(trm*=(ii+1)/(i-ii)(1/p-1)(1-p)a/(k-ii))];
              sum ] ]
```

where the input parameters are

a	=	offered traffic intensity
p	=	survival probability $e^{-\mu t}$
i	=	state of the system at $t = 0$
k	=	the index of the state the probability of which is being calculated

Note, however, that while the computational effort of (6) is constant that of the convolution algorithm is proportional to k.