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Regenerative Simulation Analysis for Loss Systems: the Single Link and Single Service Case

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Abstract. *When regenerative simulation is used for simulating blocking probabilities in loss systems, the estimator is known to be biased although strongly consistent. In this paper a method is developed for analysing the estimator's distribution as a function of increasing number of simulation cycles for the single link Erlang model. This allows us to examine in detail the effect of the choice of the regeneration state to the accuracy of the estimator in terms of its expected value and standard deviation. Some considerations on how the multiservice case can be analyzed are given as well.*

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1 Introduction

One method for simulating a multiservice loss network is the so called regenerative simulation method. The method has been developed and analyzed over the years quite thoroughly [1]-[5], but so far the analytic results are quite general in nature and, furthermore, are only asymptotic results of the type “given enough time”.

A reason to the generality of the developed theorems is that the previous research has developed theory for general regenerative stochastic processes. Thus, it has not been possible to consider certain problem specific characteristics in the development of the theory.

In our case, the problem is that of simulating the steady state distribution of the multi service loss model. In general, we know that this model rests on the assumptions that the interarrival times are exponentially distributed, but the service times may have any distribution due to the insensitivity property. Then the steady state distribution will have the following well-known form, where given $s = 1, \ldots, S$ independent traffic streams with offered traffic intensities a_s and vector $\mathbf{n} = (n_1, \ldots, n_s)$ denoting the state of the system,

$$\pi(\mathbf{n}) = \frac{p(\mathbf{n})}{G(\Omega)} \tag{1}$$

where $p(\mathbf{n})$ denotes the unnormalized probabilities

$$p(\mathbf{n}) = \prod_{s=0}^{S} \frac{a_s^{n_s}}{n_s!} e^{-a_s} \tag{2}$$

and $G(\Omega)$ is the normalizing constant calculated over the set of allowed states Ω

$$G(\Omega) = \sum_{n \in \Omega} p(\mathbf{n}) \tag{3}$$

Because the service time distribution does not affect the steady state distribution, we can choose the service times to be exponentially distributed as well. Then the problem of simulating the steady state distribution becomes that of simulating the embedded discrete time Markov chain of the multiservice loss model.

This also creates a free parameter in the simulation problem: the simulation method calls for a so called regeneration state to exist. This means that at such a state, the future evolution of the process is not dependent on the past history. In our case the simulated process has the Markov property and thus any state in the system could serve as a potential regeneration state. So, a natural question to ask is, what is the best state to choose in terms of simulation efficiency ?

As a first step in answering this question, we will here derive an analytic expression for the distribution of the estimator of the blocking probability for the single link and single traffic type case (classical Erlang model) with a given regeneration state. The expected value and standard deviation of the estimator can then be evaluated as a function of number of simulation cycles. This way we can verify the rate at which the properties of the estimator approach the known asymptotic results. As will be seen the “rate” can be significantly different depending on the starting state.

2 Regenerative Simulation

Consider the single link, single traffic type case with the following parameters:

- n = the current state of the system
- N = the size of the system (number of trunks)
- a = the offered traffic to the link

The regenerative method for simulating the blocking probability of this system is as follows: Let r denote the chosen regeneration state, and define a regenerative cycle as a path generated from the embedded DTMC, which starts from state r and ends there. Along the generated path we collect samples of two random variables: K, the number of arrivals (including blockings) in a cycle, and L, the number blockings in a cycle. The observations of these variables in m^{th} cycle are denoted as K_m and L_m. The simulation is stopped after M cycles.

Then our M-cycle estimator for the blocking probability $\hat{B}_M$ becomes

$$\hat{B}_M = \frac{1/M \sum_{m=1}^{M} L_m}{1/M \sum_{m=1}^{M} K_m} = \frac{\hat{L}_M}{\hat{K}_M} \tag{4}$$

2.1 Consistency and Bias of the Estimator

From the theory of regenerative processes (for example see [5]) we know that the real blocking probability B is completely defined by the expectations of the one cycle random variables K (arrivals) and L (blockings) as

$$B = \frac{\mathrm{E}[L]}{\mathrm{E}[K]} \tag{5}$$

In estimator (4), the numerator and denominator are by the law of large numbers both strongly consistent estimators for $\mathrm{E}[K]$ and $\mathrm{E}[L]$, i.e. $\lim_{M\to\infty} \mathrm{E}[\hat{K}_M] \to \mathrm{E}[K]$ with probability 1 (same for L). Thus, by eq. (5) the estimator (4) is also strongly consistent. However, for a finite M the estimator is biased, i.e. $\mathrm{E}[\hat{B}_M] \neq B$.

2.2 Choice of the Regeneration State

With the bias property of the estimator in mind, the next question is how does the chosen regeneration state r affect the magnitude of bias and standard deviation for a given number M of simulation cycles. In [1] it has already been shown that asymptotically the width of the obtained confidence intervals for simulations starting from different r do not depend on r. This means that given two regenerative sequences and a simulation run of length t, the two confidence intervals $I(t)$ and $I'(t)$ obtained for the two sequences have the property that $I(t)/I'(t) \to 1$ with probability 1 as $t \to \infty$.

This is only an asymptotic result and as such seems also intuitively justified, but it does not help in finding the most efficient regeneration state r when dealing with a practical simulation problem.

The regenerative simulation literature traditionally uses the heuristic of choosing the regeneration state to be the one having the smallest mean cycle length. However, in [4] it is shown that the convergence of the estimator's standard deviation is indeed affected by the choice of the regeneration state. It is also noted that the standard deviation is not necessarily minimized by this particular choice and that the question of optimal regeneration state choice remains open.

3 Analysis

3.1 Derivation of the Distribution

In [6] it was shown how the distribution of the estimator of the blocking probability could be derived for the special case, where the regeneration state was chosen such that the cycles were defined as from the blocking event to the next blocking event. Then the distribution for the estimator is only dependent on the distribution of the number of arrivals during the cycle, because the number of blockings in a cycle is exactly one. The crucial thing in the analysis was, however, that the paths had a special recursive structure, which allowed the derivation of the probability generating function (pgf) explicitly for the number of arrivals in a cycle. We will now generalize this analysis to cycles having any regeneration state.

Let K denote the number of arrivals (including the blockings) within a cycle and let L denote the number of blockings within a cycle. Also, let z be the variable in the pgf associated with arrivals, and y with blockings. The joint probability of $K = k$ and $L = l$ is given by

$$p(k,l) = \begin{cases} \Pr[K=k, L=0 \,|\!\uparrow] \Pr[\uparrow] + \Pr[K=k \,|\!\downarrow] \Pr[\downarrow] & , l = 0 \\ \Pr[K=k, L=l \,|\!\uparrow] \Pr[\uparrow] & , l > 0 \end{cases}$$

where $\uparrow$ and $\downarrow$ denote cycles which start from the regeneration state and proceed upwards and downwards respectively. Notice that when the cycle proceeds downwards from the regeneration state then we will of course not have any blockings.

The pgf for the joint probability $p(k,l)$ is defined as

$$g(z,y) = \sum_{k,l} p(k,l) z^k y^l \tag{6}$$

Now, to facilitate the analysis we will devide the analysis into two parts: 1) for cycles that proceed upwards from the regeneration state and 2) for cycles that proceed downwards, respectively. For this, we denote with $K_{\downarrow}^{n}$ and $K_{\uparrow}^{n}$ the number of arrivals during a cycle that proceeds downwards and upwards, respectively, from state n. For L there is only need for $L_{\uparrow}^{n}$ to be defined. We also introduce similar notation for the pgf:s of the random variables: $g_{\uparrow}^{n}(z,y)$ and $g_{\downarrow}^{n}(z)$ denote the pgf:s for cycles proceeding upwards or downwards from state n. Finally, we shall denote with $p_{\downarrow}^{n} = n/(a+n)$ and $p_{\uparrow}^{n} = a/(a+n)$ the probabilities for moving downwards or upwards, respectively, from state n.

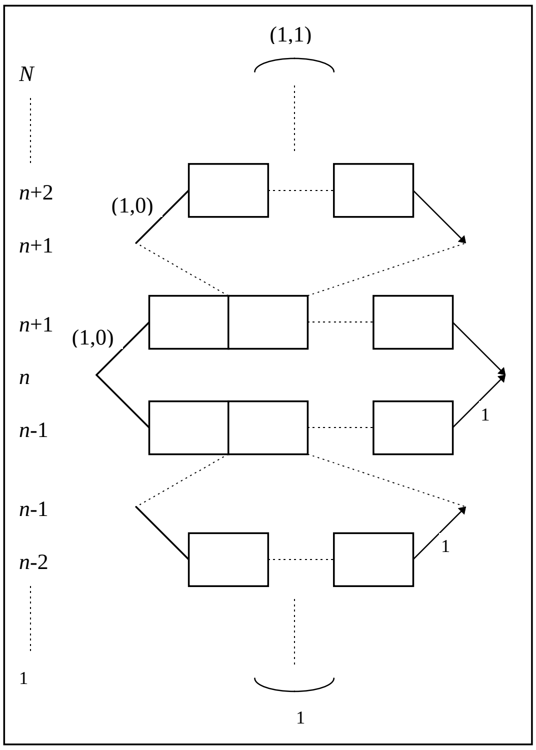

Figure 1: Regenerative cycles.

From fig.1 it can be seen that the $K_{\uparrow}^{n}$ and $L_{\uparrow}^{n}$ have a recursive structure such that for the cycles proceeding upwards from state n, the number of arrivals consists of the one coming from the transition upwards to state $n+1$ plus a random sum of arrivals from cycles beginning from and ending in state $n+1$. Now in this case, the number of such cycles, denoted with $\xi_{\uparrow}^{n+1}$, is geometrically distributed with success probability $p_{\downarrow}^{n+1}$. For the blockings the reasoning is the same, but there are not any blockings until for the cycle beginning from state N and ending there, in which case the cycle contains exactly one arrival and blocking. Then we can derive the following recursive equations for the $K_\uparrow^n$ and $L_\uparrow^n$

$$\begin{cases} (K_\uparrow^n, L_\uparrow^n) = (1,0) + \sum_{i=0}^{\xi_\uparrow^{n+1}} (K_{\uparrow,i}^{n+1}, L_{\uparrow,i}^{n+1}) \\ (K_\uparrow^N, L_\uparrow^N) = (1,1) \\ \xi_\uparrow^n \sim \text{Geom}(p_\downarrow^n) \end{cases} \tag{7}$$

Similarly we get for the number arrivals during cycles which proceed downwards (remember there are no blockings in these cycles)

$$\begin{cases} K_\downarrow^n = 1 + \sum_{i=0}^{\xi_\downarrow^{n-1}} K_{\downarrow,i}^{n-1} \\ K_\downarrow^N = 1 \\ \xi_\downarrow^n \sim \text{Geom}(p_\uparrow^n) \end{cases} \tag{8}$$

Now, in order to get the pgf for the above equations, we use the fact that for the sum of i.i.d. variables B_i with common pgf $B(z)$, $A = B_1 + \cdots + B_C$, where C itself is a random variable with pgf $C(z)$ and is independent of the B_i, has the pgf $A(z) = C(B(z))$. Specifically, in the case where $C \sim \text{Geom}(p)$, the pgf is $A(z) = \frac{p}{1-(1-p)B(z)}$. Then we will get the following recursive equations for the pgf:s

$$\begin{cases} g_\uparrow^n(z,y) = z\dfrac{p_\downarrow^{n+1}}{1 - p_\uparrow^{n+1} g_\uparrow^{n+1}(z,y)} \\ g_\downarrow^n(z) = z\dfrac{p_\uparrow^{n+1}}{1 - p_\downarrow^{n-1} g_\downarrow^{n-1}(z)} \\ g_\uparrow^N(z,y) = zy \\ g_\downarrow^1(z) = z \end{cases} \tag{9}$$

Then the complete pgf $g^n(z,y)$ for a simulation cycle starting from state n and ending there is expressed as

$$g^n(z,y) = p_\uparrow^n g_\uparrow^n(z,y) + p_\downarrow^n g_\downarrow^n(z) \tag{10}$$

Because the random variables in different regeneration cycles are i.i.d., the probability generating function for the joint probability distribution of the M-cycle estimators $\hat{K}_M$ and $\hat{L}_M$ will simply become

$$g_M^n(z,y) = (g^n(z,y))^M \tag{11}$$

3.2 Mean and Variance of the Estimator

In order to calculate the mean of the estimator (4) and its variance, we utilize some properties of the pgf. Here we simplify the notation a little bit in order to avoid confusing and overly complex notation. We simply denote with $g(z, y)$ the pgf obtained for the estimator (4) through equations (9) - (11), i.e. we omit the dependendence on M and n. Also, we denote with L (blockings) and K (arrivals), the random variables associated with the variables z and y in the pgf, respectively. $g^{(i,j)}(z, y)$ is used to denote the i^{th} and j^{th} partial derivatives with respect to z and y.

Then for the numerator part, i.e. random variable L, we shall need the following relations

$$\begin{aligned}
\mathrm{E}\left[L\right] &= \sum_{k,l} l\ p(k,l) = \frac{\partial}{\partial y}\left(\sum_{k,l} p(k,l) z^k y^l\right)\Bigg|_{z=1,y=1} \\
&= g^{(0,1)}(1,1) \qquad (12) \\
\mathrm{E}\left[L^2\right] &= \sum_{k,l} l^2\ p(k,l) = \sum_{k,l} l(l-1)\ p(k,l) + \sum_{k,l} l\ p(k,l) \\
&= \frac{\partial}{\partial y^2}\left(\sum_{k,l} p(k,l) z^k y^l\right)\Bigg|_{z=1,y=1} + \frac{\partial}{\partial y}\left(\sum_{k,l} p(k,l) z^k y^l\right)\Bigg|_{z=1,y=1} \\
&= g^{(0,2)}(1,1) + g^{(0,1)}(1,1) \qquad (13)
\end{aligned}$$

For calculating the statistics of the arrivals K, we need to be able to evaluate the following

$$\begin{aligned}
\mathrm{E}\left[\frac{1}{K}\right] &= \sum_k \frac{1}{k} \sum_l p(k,l) \\
&= \int_0^1 \frac{g(z,1)}{z}\ dz \qquad (14) \\
\mathrm{E}\left[\frac{1}{K^2}\right] &= \sum_k \frac{1}{k^2} \sum_l p(k,l) \\
&= \int_0^1 \frac{1}{z} \int_0^z \frac{g(u,1)}{u}\ du\ dz \\
&= \int_0^1 \frac{1}{u}\ du \int_u^1 \frac{1}{z} g(u,1)\ dz \\
&= -\int_0^1 \frac{\ln u}{u} g(u,1)\ du \qquad (15)
\end{aligned}$$

Where in equation (15) we have changed the order of integration to simplify the integration to a single integral.

Equation (14) can be proved quite easily. The pgf of the arrivals K is derived from the total pgf $g(z,y)$ as $g(z,1) = \sum_k z^k \sum_l p(k,l) = \sum_k p^*(k) z^k$, where we have denoted with $p^*(k)$ the marginal probabilities of K. Also, note that $p^*(0) = 0$. Then we have

$$\begin{aligned} \int_0^1 \frac{g(z,1)}{z}\, dz &= \int_0^1 \left(\sum_{k=1}^{\infty} \frac{1}{k} p^*(k) z^{k-1} \right) dz \\ &= \Big/_0^1 \left(p^*(1) z + \frac{1}{2} p^*(2) z^2 + \cdots + \frac{1}{n} p^*(n) z^n + \cdots \right) \\ &= \sum_{k=1}^{\infty} \frac{1}{k}\, p^*(k) \end{aligned}$$

Eq. (15) can be proved in a similar fashion by only doing the integration twice. Then by using methods as in equations (12) - (15) we can derive the following

$$\mathrm{E}\left[\hat{B}_M\right] = \mathrm{E}\left[\frac{L}{K}\right] = \int_0^1 \left(\frac{1}{z} g^{(0,1)}(z,1) \right) dz \tag{16}$$

$$\mathrm{E}\left[\hat{B}_M^2\right] = \mathrm{E}\left[\frac{L^2}{K^2}\right] = -\int_0^1 \left(\frac{\ln z}{z} g^{(0,2)}(z,1) + g^{(0,1)}(z,1) \right) dz \tag{17}$$

Now, by using equations (16) and (17) we are able to calculate the mean and variance of the distribition for estimator (4) after M simulation cycles.

3.3 Comparing Results for Different Regeneration States

To compare the effect of different regeneration states, we must be able to compare the mean and variance of the estimator for equal number of generated events from the DTMC of the process. However, equations (16) and (17) only give us values as a function of increasing number of simulated cycles.

First we can notice that the mean number of arrivals in a cycle is given by

$$\mathrm{E}[K] = g^{(1,0)}(1,1) \tag{18}$$

Also, we know that the number of departures during a cycle must equal the number of arrivals. Thus, it is sufficient to compare the efficiency of the estimator for different starting states as a function of the number of arrivals needed to acheive certain precision. Therefore, when comparing the estimators we just need to evaluate (16) and (17) until a sufficient number of cycles, so that for each regeneration state, the mean number of arrivals needed to achieve this accuracy is enough.

Also, we define a metric $ERR_{\hat{B}_M}$ that tries to combine the effect of both the bias and the influence of the standard deviation as follows

$$
\begin{aligned}
ERR_{\hat{B}_M} &= \sqrt{\mathrm{E}\left[\left(\hat{B}_M - B\right)^2\right]} \\
&= \sqrt{\left(\mathrm{E}\left[\hat{B}_M\right] - B\right)^2 + \sigma^2_{\hat{B}_M}}
\end{aligned}
\tag{19}
$$

4 Numerical Results

In this section we present the results of using equations (9) - (11) and then evaluating equations (16) and (17) on a system with $N = 6$ and $a = 2$. The real blocking probability is $\mathrm{Erl}(2, 6) = 0.0121$ in this case. We have also included in the figures the results for choosing the regeneration state as in [6], i.e. from a blocking event to the next blocking event.

In the following graphs, fig. 2 - 4, we have plotted the results for all possible regeneration states for increasing number of simulation cycles. In all the graphs, the lowest curve corresponds to the results for having the regeneration state as state 1, and the next upper curve corresponds to state 2 etc.

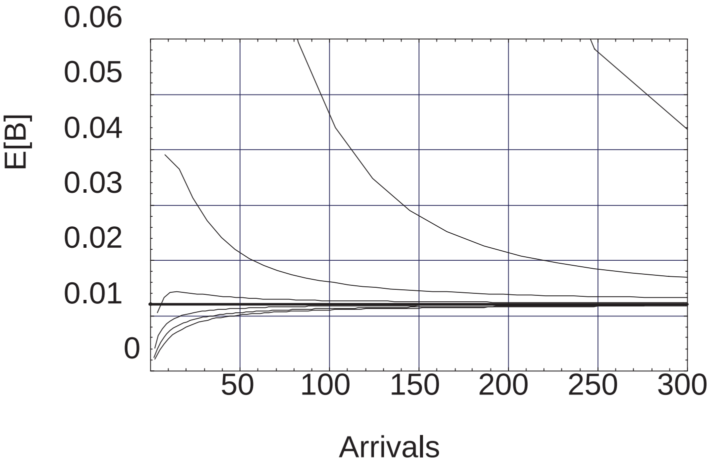

Figure 2: Expected value of the estimator.

In fig. 2 the horizontal line represents the true value of the blocking probability.

The curves clearly show that at least one should not choose any state as the regeneration state. On the other hand they show rapid convergence as the number of arrivals increase, i.e. simulation time, showing good agreement with the asymptotic results. What is perhaps most surprising in the curves is that the standard deviation $\sigma_{\hat{B}_M}$ increases when the chosen regeneration state becomes greater. Also, it can be seen that with the combined metric eq. (19), the standard deviation is the one dominating the graph behavior.

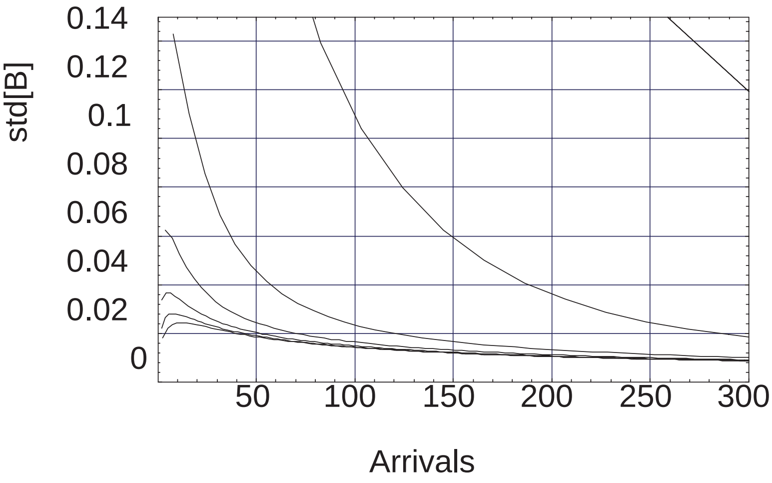

Figure 3: Standard deviation of the estimator.

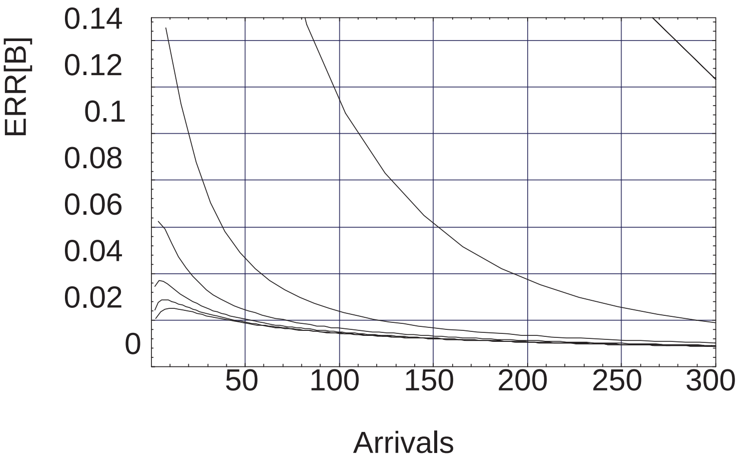

Figure 4: $ERR[\hat{B}]$ metric of the estimator.

Thus from fig. 4, state 1 would be the best choice for the regeneration state. However, the mean number of arrivals for cycles beginning from the different states was for this example $\mathrm{E}[K] = [2.34\ 1.84\ 2.21\ 3.68\ 7.88\ 20.69\ 82.75]$, where the last number corresponds to cycles from a blocking event to the next. But, from the graph it can be seen that for all the states 1-4, the standard deviation drops to almost equal level very quickly. On the other hand, from fig.2 one can see that for all states 1-3, the estimate would be biased downwards, but the estimate for state 4 would remain biased upwards. So a more conservative choice for the regeneration state would be to choose state 4.

5 Conclusions

In this paper a method was presented for analyzing the estimators behavior when simulating the blocking probability for a single link and single traffic type case using the regenerative simulation method. A recursive presentation for the probability generating function for the arrivals and blockings during a regenerative cycle was derived, see eqs. (9) - (11). Also, the necessary formulas for calculating the expected value and standard deviation for the ratio type estimator were given, eqs. (16) and (17).

The numerical results showed that the regeneration state should be chosen quite near the state having the shortest mean cycle length. However, there is no clear choice for the best state, because it depends on how much weight is given to the magnitude of the bias of the estimator and its standard deviation. More research is needed to determine this relationship more accurately.

The extension of this analysis approach in the multi service case leads to certain problems. The main problem is that we no longer have the situation where the number of “sub cycles” within a cycle would be geometrically distributed. This is a result of the fact that now the paths can wander in the state space in different directions, and as a consequence the nice regular properties of the paths in the one dimensional state space are destroyed.

One possible approach to the problem would be to change the recursions to follow the evolution of the system from arrival to another. However, this approach leads to a situation, where we can no longer calculate the probability generating function explicitly. Instead we will have a set of coupled equations relating the pgf of each state in the system to one another. For the purpose of doing numerical calculations it is not a problem, because we can always calculate the solutions at any fixed point, but we will not be able to get an analytical expression for the pgf anymore.

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