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Handbook of FBM formulae

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Mathematical functions

Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

$$\Gamma(z+1) = z\Gamma(z) \qquad \Gamma(1) = \Gamma(2) = 1$$

Beta function

$$B(\mu,\nu) = \int_{n=0}^1 x^{\mu-1}(1-x)^{\nu-1} dx = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}$$

The Gauss hypergeometric function

$$F(a,b,c,z) \;=\; \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

Integral relations

$$\int_0^1 t^{\mu-1}(1-t)^{\nu-1}(c-t)^{-\mu-\nu} \mathrm{d}t \;=\; c^{-\nu}(c-1)^{-\mu} B(\mu,\nu) \qquad \mu,\nu > 0,\ c > 1$$

$$\int_1^c t^{\mu}(t-1)^{\nu} \mathrm{d}t \;=\; \int_0^{1-1/c} s^{\nu}(1-s)^{-\mu-\nu-2} \mathrm{d}s \qquad \mu \in \mathbb{R},\ \nu > -1,\ c > 1$$

$$\int_0^1 t^{\mu-1}(1-t)^{\nu-1}(c-t)^{-\mu-\nu+1} \mathrm{d}t \;=\; (\mu+\nu-1)B(\mu,\nu)c^{-\nu+1} \cdot \int_0^1 s^{\mu+\nu-2}(c-s)^{-\mu} \mathrm{d}s \qquad \mu,\nu > 0,\ \mu+\nu > 1,\ c > 1$$

$$\int_0^1 t^{-\alpha}(1-t)^{-\alpha}|x-t|^{2\alpha-1} \mathrm{d}t \;=\; B(1-\alpha,\alpha) \qquad \alpha \in (0,\tfrac{1}{2}),\ x \in (0,1)$$

Fractional Brownian motion Z_t

- Z_t has stationary increments
- $Z_0 = 0$, and $\mathrm{E}[Z_t] = 0$ for all t
- $\mathrm{E}[Z_t^2] = |t|^{2H}$ for all t
- Z_t is Gaussian
- Z_t has continuous sample paths

Self-similarity parameters

$$\begin{aligned} H &= \text{the Hurst parameter} \\ \alpha &= H - \frac{1}{2} \end{aligned}$$

For $H = 1/2$ ($\alpha = 0$) Z_t is identical to the standard brownian motion W_t.

Useful constants

$$c = \frac{1}{B(1+\alpha, 1-\alpha)} = \frac{\sin \alpha\pi}{\alpha\pi}$$

$$C = \sqrt{\frac{(1+2\alpha)}{\alpha B(\alpha, 1-2\alpha)}} = \sqrt{\frac{(1+2\alpha)\Gamma(1-\alpha)}{\Gamma(1-2\alpha)\Gamma(1+\alpha)}}$$

$$C' = \frac{c}{C}$$

$$\lim_{\alpha\to 0} c = \lim_{\alpha\to 0} C = \lim_{\alpha\to 0} C' = 1$$

Covariances

$$\begin{aligned} \mathrm{Cov}[dZ_t, dZ_s] &= H(2H-1)|t-s|^{2H-2} dt\, ds \\ &= \alpha(1+2\alpha)|t-s|^{2\alpha-1} dt\, ds \\ &\stackrel{\text{def}}{=} r(t,s) dt ds \end{aligned}$$

$$\mathrm{Cov}[Z_t, Z_s] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

$$
\begin{aligned}
&= \frac{1}{2}(t^{1+2\alpha} + s^{1+2\alpha} - |t-s|^{1+2\alpha}) \\
&\stackrel{\text{def}}{=} R(t,s)dtds
\end{aligned}
$$

$$
\mathrm{Cov}[W_t, W_s] = \min(t,s)
$$

Consistently with $\mathrm{Cov}[Z_t, Z_s] = \int_0^t du \int_0^s dv \mathrm{Cov}[dZ_u, dZ_v]$ we have the elementary identity

$$
\alpha(1+2\alpha)\int_0^t du \int_0^s dv |u-v|^{2\alpha-1} = \frac{1}{2}(t^{1+2\alpha} + s^{1+2\alpha} - |t-s|^{1+2\alpha}) \quad \alpha > 0
$$

Limit form

$$
\lim_{\alpha \to 0} \alpha(1+2\alpha)|t-s|^{2\alpha-1} = \delta(t-s) \qquad \text{(Dirac's delta function)}
$$

Covariance kernels

The following definitions apply for $t \geq s$. For $t < s$ the kernels can be defined to be zero.

$$
\begin{aligned}
k(t,s) &= \alpha\,(\frac{t}{s})^{\alpha}(t-s)^{\alpha-1} \\
K(t,s) &= \int_s^t k(u,s)du \\
&= \alpha\, s^{-\alpha}\int_s^t u^{\alpha}(u-s)^{\alpha-1}du \\
&= (t-s)^{\alpha}\, F(-\alpha, \alpha, 1+\alpha, 1-\frac{t}{s}) \\
K_W(t,s) &= s^{-\alpha}\Big(t^{\alpha}(t-s)^{-\alpha} - \alpha\int_s^t u^{\alpha-1}(u-s)^{-\alpha}du\Big) \\
&= s^{-\alpha}\Big((\frac{t}{t-s})^{\alpha} - \frac{\alpha}{1-\alpha}(\frac{t}{s}-1)^{1-\alpha}F(1-\alpha, 1-\alpha, 2-\alpha, 1-\frac{t}{s})\Big)
\end{aligned}
$$

where F is the Gauss hypergeometric function.

$$
\alpha(1+2\alpha)|t-s|^{2\alpha-1} = C^2\int_0^{\min(t,s)} k(t,u)k(s,u)\,du
$$

$$
\begin{aligned}
\frac{1}{2}(t^{1+2\alpha} + s^{1+2\alpha} - |t-s|^{1+2\alpha}) &= C^2 \int_0^{\min(t,s)} K(t,u)K(s,u)\,du \\
\min(t,s) &= C'^2 \int_0^t du \int_0^s dv K_W(t,u)K_W(s,v)r(u,v)
\end{aligned}
$$

Asymptotic forms

$$
\begin{aligned}
& & k(t,s)|_{s\to 0} &\simeq \alpha\, t^{2\alpha-1}s^{-\alpha} \\
k(t,s)|_{t\to\infty}^{t-s=\text{const.}} &\simeq & k(t,s)|_{s\to t} &\simeq \alpha\,(t-s)^{\alpha-1} \\
& & K(t,s)|_{s\to 0} &\simeq \frac{1}{2}\, t^{2\alpha}s^{-\alpha} \\
K(t,s)|_{t\to\infty}^{t-s=\text{const.}} &\simeq & K(t,s)|_{s\to t} &\simeq (t-s)^{\alpha} \\
& & \lim_{\alpha\to 0} K(t,s) &= 1 \\
K_W(t,s)|_{t\to\infty}^{t-s=\text{const.}} &\simeq & K_W(t,s)|_{s\to t} &\simeq (t-s)^{\alpha}
\end{aligned}
$$

Integral representations

Define

$$
\begin{cases}
\tilde{Z}_t = \int_0^t s^{-\alpha} dZ_s \\
\tilde{W}_t = \int_0^t s^{-\alpha} dW_s
\end{cases}
\qquad \text{i.e.} \qquad
\begin{cases}
Z_t = \int_0^t s^{\alpha} d\tilde{Z}_s \\
W_t = \int_0^t s^{\alpha} d\tilde{W}_s
\end{cases}
$$

The following mutual representations hold:

$$
\begin{aligned}
\tilde{Z}_t &= C\int_0^t (t-s)^{\alpha} d\tilde{W}_s &= C\int_0^t s^{-\alpha}(t-s)^{\alpha} dW_s \\
\tilde{W}_t &= C'\int_0^t (t-s)^{-\alpha} d\tilde{Z}_s &= C'\int_0^t s^{-\alpha}(t-s)^{-\alpha} dZ_s
\end{aligned}
$$

$$
\begin{aligned}
Z_t &= C\int_0^t K(t,s)\,dW_s \\
W_t &= C'\int_0^t K_W(t,s)\,dZ_s
\end{aligned}
$$

By a) applying the above representation for Z_t to two time instants t and s ($s < t$), b) letting $t \to \infty$ while keeping the difference $t - s$ constant, c) using the asymptotic form of

$K(t,s)$ and d) shifting the origin to s, we get the Mandelbrot and Van Ness representation

$$Z_t = C\Big(\int_{-\infty}^{t}(t-u)^\alpha dW_u - \int_{-\infty}^{0}(-u)^\alpha dW_u\Big)$$

Prediction formula

$$\begin{aligned}\mathrm{E}\left[Z_T \,|\, Z_s \in [0,t]\right] &= C\int_0^t K(T,s)dW_s \\ &= Z_t + \int_0^t dZ_u\,\Psi_T(t,u)\end{aligned}$$

where (with $K^{(0,1)}(\cdot,\cdot)$ denoting the derivative with respect to the second argument)

$$\begin{aligned}\Psi_T(t,u) &= c\,\Big(K(T,t)K_W(t,u) - \int_u^t K^{(0,1)}(T,s)K_W(s,u)ds\Big) - 1 \\ &= \frac{\sin\alpha\pi}{\pi}u^{-\alpha}(t-u)^{-\alpha}\int_t^T \frac{s^\alpha(s-t)^\alpha}{s-u}ds\end{aligned}$$

Conditional distributions

Let $\mathbf{z} = (Z_{t_1},\ldots,Z_{t_k})^T$ be a k-vector of values of the process Z_t. $\mathbf{z}$ is a multivariate Gaussian vector whose pdf is given by

$$f(\mathbf{z}) = \frac{1}{(2\pi)^{k/2}|\mathbf{\Gamma}|^{1/2}}e^{-\frac{1}{2}\mathbf{z}^T\mathbf{\Gamma}^{-1}\mathbf{z}},$$

where $\mathbf{\Gamma} = \mathrm{E}[\mathbf{z}\,\mathbf{z}^T]$ is the (symmetric) covariance matrix

$$\Gamma_{i,j} = \mathrm{Cov}[Z_{t_i}, Z_{t_j}] = \mathrm{E}[Z_{t_i}Z_{t_j}] \qquad i,j = 1,\ldots,k$$

Unconditional values of $\mathbf{z}$ can be generated with the aid of the representation

$$\mathbf{z} = \mathbf{\Gamma}^{1/2}\mathbf{w}$$

where $\mathbf{w} = (w_1,\ldots,w_k)^T$ is a k-vector of independent $N(0,1)$-distributed Gaussian variables. (Note that the previous integral representation for Z_t is a continuous counterpart of this relation.)

Consider now partitioning of $\mathbf{z}$ into two parts $\mathbf{z}_1$ and $\mathbf{z}_2$ with dimensions m and $n = k-m$,

$$\mathbf{z} = (\underbrace{Z_{t_1},\ldots,Z_{t_m}}_{\mathbf{z}_1^T},\underbrace{Z_{t_{m+1}},\ldots,Z_{t_k}}_{\mathbf{z}_2^T})$$

Denote $\mathbf{A} = \mathbf{\Gamma}^{-1}$ and let similarly

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

be the partitioning of $\mathbf{A}$ into $m \times m$, $m \times n$, $n \times m$, and $n \times n$ submatrices, $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$ and $\mathbf{A}_{22}$ with $\mathbf{A}_{12} = \mathbf{A}_{21}^T$. The conditional distribution of $\mathbf{z}_1$, given $\mathbf{z}_2$, is Gaussian with mean and covariance

$$\begin{aligned} \mathrm{E}\,[\mathbf{z}_1 \,|\, \mathbf{z}_2] &= -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{z}_2 \\ \mathrm{E}\,[\mathbf{z}_1\mathbf{z}_1^T \mid \mathbf{z}_2] &= \mathbf{A}_{11}^{-1} \end{aligned}$$

Values of $\mathbf{z}_1$ can be generated with the aid of the representation in terms of an n-dimensional Gaussian vector $\mathbf{w}_1$

$$\mathbf{z}_1 = -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{z}_2 + \mathbf{A}_{11}^{-1/2}\mathbf{w}_1$$

In particular, for $m = n = 1$ we get the explicit forms

$$\begin{aligned} \mathrm{E}\,[Z_s \,|\, Z_t] &= \frac{\mathrm{Cov}[Z_s, Z_t]}{\mathrm{Var}[Z_t]} \;=\; h(\frac{s}{t})Z_t \\ \mathrm{V}[Z_s \,|\, Z_t] &= \Big(1 - \frac{\mathrm{Cov}[Z_s, Z_t]^2}{\mathrm{Var}[Z_s]\mathrm{Var}[Z_t]}\Big)\,\mathrm{Var}[Z_s] \\ &= (1 - h(\frac{s}{t})h(\frac{t}{s}))\, s^{2H} \end{aligned}$$

where $h(x)$ stands for the function

$$h(x) = \frac{1}{2}(1 + x^{2H} - |1 - x|^{2H})$$

with properties

$$\begin{aligned} h(x) &= x^{2H}h(\frac{1}{x}) \\ h'(x) &= \begin{cases} H(x^{2H-1} + (1-x)^{2H-1}) & x \le 1 \\ H(x^{2H-1} - (x-1)^{2H-1}) & x > 1 \end{cases} \end{aligned}$$

As an immediate corollary we have

$$\begin{aligned} \mathrm{E}\,[dZ_s \,|\, Z_t] &= \frac{ds}{t}h'(\frac{s}{t})Z_t \\ \mathrm{E}\left[\int_a^b f(s)dZ_s \,|\, Z_t\right] &= \frac{Z_t}{t}\int_a^b h'(\frac{s}{t})f(s)ds \end{aligned}$$

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