Helsinki University of Technology
Signal Processing Laboratory

S-38.411 Signal Processing in Telecommunications I

Spring 2000

Lecture 6: Adaptive equalizers 1

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Timetable

L1 Introduction; models for channels and communication systems
L2 Channel capacity
L3 Transmit and receive filters for bandlimited AWGN channels
L4 Optimal linear equalizers for linear channels 1
L5 Optimal linear equalizers for linear channels 2
L6 Adaptive equalizers 1
L7 Adaptive equalizers 2
L8 Nonlinear receivers 1: DFE equalizers
L9 Nonlinear receivers 2: Viterbi algorithm
L10 GL1: DSP for Fixed Networks / *Matti Lehtimäki, Nokia Networks*
L11 GL2: DSP for Digital Subscriber Lines / *Janne Väänänen, Tellabs*
L12 GL3: DSP for CDMA Mobile Systems / *Kari Kalliojärvi, NRC*
L13 Course review, questions, feedback
E 24.5. (Wed) 9-12 S4 **Exam**

Contents of Lecture 6

Adaptive equalizers 1

I. MSE Gradient algorithm

II. Convergence analysis of MSEG algorithm

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I. MSE Gradient algorithm

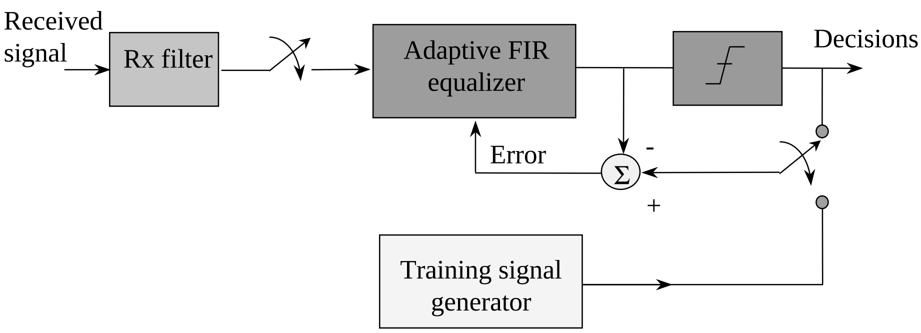
MSE gradient algorithm

- In the previous lecture, we derived solution for optimal MMSE FIR equalizer
- Requires knowledge of channel in form of Rx signal autocorrelation matrix and crosscorrelation vector + matrix inversion
- Difficult to implement in practice, because the channel is unknown and often time varying
- In this lecture:

  Construct adaptive algorithm that can learn the channel automatically and adapt to changing conditions

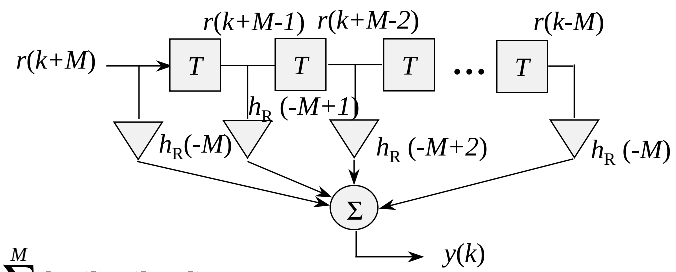
MSE gradient algorithm...

- System model:



MSE gradient algorithm...

- Adaptive FIR filter structure:



$$y(k) = \sum_{l=-M}^{M} h_R(l) r(k-l)$$

MSE gradient algorithm...

- Recall the MSE form of the FIR solution:

$$\mathrm{E}\left[e^2(k)\right] = E_{\mathrm{MIN}} + \left(\mathbf{h}_{\mathrm{R}} - \mathbf{h}_{\mathrm{R,opt}}\right)^{\mathrm{T}} \mathbf{R}\left(\mathbf{h}_{\mathrm{R}} - \mathbf{h}_{\mathrm{R,opt}}\right)$$

- Properties:
  - R is positive definite, with Toeplitz structure
  - error surface is *hyperparaboloid* with a unique minimum.

MSE gradient algorithm...

- General form of gradient algorithm:

$$\mathbf{h}_{\mathrm{R}}[j+1] = \mathbf{h}_{\mathrm{R}}[j] - \frac{\beta}{2}\nabla_{\mathbf{h}_{\mathrm{R}}}\mathrm{E}\left[e^2(k)\right]$$

- Differentiate MSE:

$$\nabla_{\mathbf{h}_{\mathrm{R}}}\mathrm{E}\left[e^2(k)\right] = 2\mathbf{R}\mathbf{h}_{\mathrm{R}} - 2\mathbf{p}$$

MSE gradient algorithm...

- MSE gradient algorithm by insertion:

$$\begin{aligned}\mathbf{h}_{\mathrm{R}}[j+1] &= \mathbf{h}_{\mathrm{R}}[j] - \frac{\beta}{2}\left(2\mathbf{R}\mathbf{h}_{\mathrm{R}}[j] - 2\mathbf{p}\right)\\ &= \mathbf{h}_{\mathrm{R}}[j] + \beta\left(\mathbf{p} - \mathbf{R}\mathbf{h}_{\mathrm{R}}[j]\right)\end{aligned}$$

- Alternative form:

$$\mathbf{h}_{\mathrm{R}}[j+1] = (\mathbf{I} - \beta\mathbf{R})\mathbf{h}_{\mathrm{R}}[j] + \beta\mathbf{p}$$

- Note! "*j*" is iteration index, independent of time!

II. Convergence analysis of MSEG algorithm

Convergence analysis of MSEG

- General MSEG algorithm:

$$\mathbf{h}_{\mathrm{R}}[j+1] = (\mathbf{I} - \beta\mathbf{R})\mathbf{h}_{\mathrm{R}}[j] + \beta\mathbf{p}$$

- Let us study adaptive 1-tap FIR filter:

$$h_{\mathrm{R}}[j+1] = (1 - \beta R_0)h_{\mathrm{R}}[j] + \beta p_0$$

- Subtract optimum solution from both sides:

$$\begin{aligned}(h_{\mathrm{R}}[j+1] - h_{\mathrm{R,opt}}) &= (1-\beta R_0)(h_{\mathrm{R}}[j] - h_{\mathrm{R,opt}}) \\ &= (1-\beta R_0)^j (h_{\mathrm{R}}[0] - h_{\mathrm{R,opt}}).\end{aligned}$$

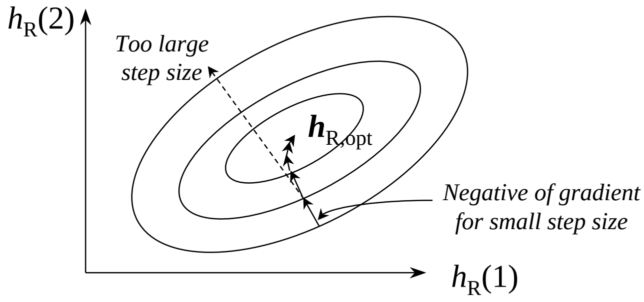
Convergence analysis of MSEG...

- The solution converges to optimum if and only if:

$$\left|1-\beta R_0\right|<1$$
$$\Leftrightarrow\ 0<\beta<\frac{2}{R_0}.$$

- Upper limit for convergence parameter!

Convergence analysis of MSEG...



- Illustration of 2-tap filter convergence

Convergence analysis of MSEG...

- General N-tap adaptive FIR filter:

$$\left(\mathbf{h}_{\mathrm{R}}[j+1]-\mathbf{h}_{\mathrm{R,opt}}\right)=(\mathbf{I}-\beta\mathbf{R})^{j}\left(\mathbf{h}_{\mathrm{R}}[0]-\mathbf{h}_{\mathrm{R,opt}}\right).$$

- How to analyze convergence?
- Spectral decomposition:

$$\mathbf{R}=\sum_{i=1}^{N}\lambda_i \mathbf{v}_i \mathbf{v}_i^{\mathrm{H}}$$

- Eigenvalues & eigenvectors!

Convergence analysis of MSEG...

- Modal decomposition for adaptive equation matrix:

$$(\mathbf{I}-\beta\mathbf{R})^{j}=\sum_{i=1}^{N}(1-\beta\lambda_i)^{j}\mathbf{v}_i \mathbf{v}_i^{\mathrm{H}}$$

- Convergence criterion for N modes:

$$|1-\beta\lambda_i|<1, \quad i=1,2,\ldots, N$$

- Criterion for the largest eigenvalue:

$$|1-\beta\lambda_{\mathrm{MAX}}|<1 \quad \Leftrightarrow \quad 0<\beta<\frac{2}{\lambda_{\mathrm{MAX}}}.$$

Convergence analysis of MSEG...

- Optimum choice for step parameter (fastest convergence):

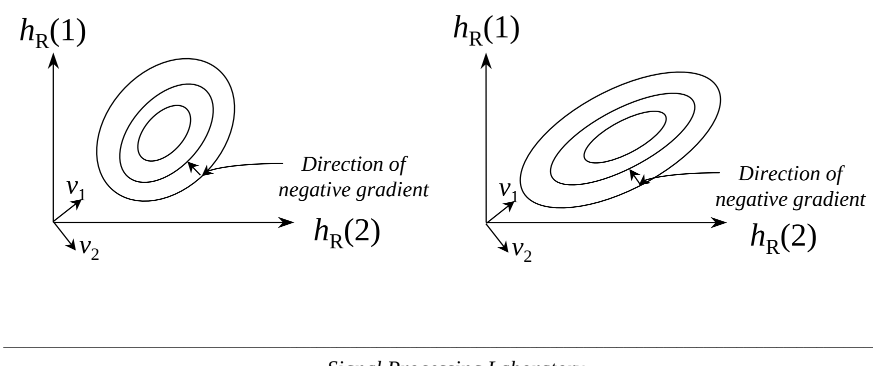
$$\beta_{\mathrm{OPT}} = \frac{2}{\lambda_{\mathrm{MIN}} + \lambda_{\mathrm{MAX}}}.$$

- Convergence is slow for large eigenvalue spread, i.e.

$$\lambda_{\mathrm{MAX}} / \lambda_{\mathrm{MIN}} >> 1$$

Convergence analysis of MSEG...

- Illustration of error surfaces for small and large (right) eigenvalue spread:



Convergence analysis of MSEG...

- Relation of eigenvalues and Rx signal power spectrum:

$$\underset{f}{\text{MIN}}[S_r(f)] \leq \lambda \leq \underset{f}{\text{MAX}}[S_r(f)]$$

Summary

Today we discussed:

Adaptive equalizers 1
I. MSE Gradient algorithm
II. Convergence analysis of MSEG algorithm

Next week: Adaptive equalizers 2:
Stochastic gradient (or LMS) algorithm