Lecture 6: Adaptive equalizers 1

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Timetable

L1 Introduction; models for channels and communication systems
L2 Channel capacity
L3 Transmit and receive filters for bandlimited AWGN channels
L4 Optimal linear equalizers for linear channels 1
L5 Optimal linear equalizers for linear channels 2
L6 Adaptive equalizers 1
L7 Adaptive equalizers 2
L8 Nonlinear receivers 1: DFE equalizers
L9 Nonlinear receivers 2: Viterbi algorithm
L10 GL1: DSP for Fixed Networks / Matti Lehtimäki, Nokia Networks
L11 GL2: DSP for Digital Subscriber Lines / Janne Vääntänen, Tellabs
L12 GL3: DSP for CDMA Mobile Systems / Kari Kallooriäri, NRC
L13 Course review, questions, feedback
E 24.5. (Wed) 9-12 S4 Exam
Contents of Lecture 6

Adaptive equalizers 1
I. MSE Gradient algorithm
II. Convergence analysis of MSEG algorithm

I. MSE Gradient algorithm
MSE gradient algorithm

- In the previous lecture, we derived solution for optimal MMSE FIR equalizer
- Requires knowledge of channel in form of Rx signal autocorrelation matrix and crosscorrelation vector + matrix inversion
- Difficult to implement in practice, because the channel is unknown and often time varying
- In this lecture:  
  *Construct adaptive algorithm that can learn the channel automatically and adapt to changing conditions*

MSE gradient algorithm...

- System model:
MSE gradient algorithm...

- Adaptive FIR filter structure:

\[
y(k) = \sum_{l=-M}^{M} h_R(l) r(k-l)
\]

- Recall the MSE form of the FIR solution:

\[
E[e^2(k)] = E_{\text{MIN}} + (h_R - h_{R_{\text{opt}}})^T R(h_R - h_{R_{\text{opt}}})
\]

- Properties:
  - \( R \) is positive definite, with Toeplitz structure
  - error surface is hyperparaboloid with a unique minimum.
MSE gradient algorithm...

◆ General form of gradient algorithm:

\[ h_R[j + 1] = h_R[j] - \frac{\beta}{2} \nabla_{h_R} E[e^2(k)] \]

◆ Differentiate MSE:

\[ \nabla_{h_R} E[e^2(k)] = 2R h_R - 2p \]

MSE gradient algorithm...

◆ MSE gradient algorithm by insertion:

\[ h_R[j + 1] = h_R[j] - \frac{\beta}{2} (2R h_R[j] - 2p) \]

\[ = h_R[j] + \beta(p - R h_R[j]) \]

◆ Alternative form:

\[ h_R[j + 1] = (I - \beta R) h_R[j] + \beta p \]

◆ Note! “j” is iteration index, independent of time!
II. Convergence analysis of MSEG algorithm

Convergence analysis of MSEG

- General MSEG algorithm:
  \[ h_R[j + 1] = (1 - \beta R_0) h_R[j] + \beta p \]

- Let us study adaptive 1-tap FIR filter:
  \[ h_R[j + 1] = (1 - \beta R_0) h_R[j] + \beta p_0 \]

- Subtract optimum solution from both sides:
  \[ (h_R[j + 1] - h_{R,\text{opt}}) = (1 - \beta R_0) (h_R[j] - h_{R,\text{opt}}) \]
  \[ = (1 - \beta R_0) (h_R[0] - h_{R,\text{opt}}) \]
Convergence analysis of MSEG...

- The solution converges to optimum if and only if:

\[ |1 - \beta R_0| < 1 \]
\[ \iff 0 < \beta < \frac{2}{R_0}. \]

- Upper limit for convergence parameter!

Illustration of 2-tap filter convergence
Convergence analysis of MSEG...

- General $N$-tap adaptive FIR filter:

$$\left( h_R[j+1] - h_{R,\text{opt}} \right) = (I - \beta R)' \left( h_R[0] - h_{R,\text{opt}} \right).$$

- How to analyze convergence?
- Spectral decomposition:

$$R = \sum_{i=1}^{N} \lambda_i v_i v_i^H$$

- Eigenvalues & eigenvectors!

Convergence analysis of MSEG...

- Modal decomposition for adaptive equation matrix:

$$(I - \beta R)' = \sum_{i=1}^{N} (1 - \beta \lambda_i) v_i v_i^H$$

- Convergence criterion for $N$ modes:

$$\left| 1 - \beta \lambda_i \right| < 1, \quad i = 1, 2, ..., N$$

- Criterion for the largest eigenvalue:

$$\left| 1 - \beta \lambda_{\text{MAX}} \right| < 1 \quad \Rightarrow \quad 0 < \beta < \frac{2}{\lambda_{\text{MAX}}}.$$
Convergence analysis of MSEG...

- Optimum choice for step parameter (fastest convergence):
  \[ \beta_{\text{opt}} = \frac{2}{\lambda_{\text{MIN}} + \lambda_{\text{MAX}}} \]

- Convergence is slow for large eigenvalue spread, i.e.
  \[ \frac{\lambda_{\text{MAX}}}{\lambda_{\text{MIN}}} >> 1 \]

Illustration of error surfaces for small and large (right) eigenvalue spread:
Convergence analysis of MSEG...

- Relation of eigenvalues and Rx signal power spectrum:

\[ \min_{\lambda} S_r(f) \leq \lambda \leq \max_{\lambda} S_r(f) \]

Summary

Today we discussed:

Adaptive equalizers 1
I. MSE Gradient algorithm
II. Convergence analysis of MSEG algorithm

Next week: Adaptive equalizers 2:
Stochastic gradient (or LMS) algorithm