S-38.411 Signal Processing in Telecommunications I
Spring 2000
Lecture 3: Transmit and receive filters for bandlimited
AWGN channels

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Timetable

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II. Nyquist criterion
III. Matched filter
IV. Root-Nyquist filtering
V. Tx and Rx filters in practice

I. Ideal sinc solution
Ideal sinc solution

- In the previous lecture, we discussed the basic limits for data transmission in practical channels.
- In order to achieve the capacity, the power spectrum \( S_x(f) \) of the transmitted signal must be chosen right:

\[
S_{x,\text{opt}}(f) = \begin{cases} 
L - \frac{S_x(f)}{|C(f)|^2}, & f \in F \\
0, & \text{elsewhere}
\end{cases}
\]

- How can we design signals with desired power spectrum?
- How should the receiver (pre)process the received signal for detection (estimation) of transmitted data symbols?

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Ideal sinc solution...

- System model:

\[
\sum_{k=0}^{n} a_k \delta(t-kT) h_f(t) \delta(t) = x(t) + n(t)
\]

\[
r(t) = x(t) + n(t)
\]

\[
z_k = \hat{\alpha}_k \hat{\delta}_k
\]
Ideal sinc solution...

- Structure of transmitted signal with *linear modulation* methods (PAM, QAM, MPSK, etc.):

\[
x(t) = h_T(t) \star \sum_{k=-\infty}^{\infty} a_k \delta(t - kT) = \sum_{k=-\infty}^{\infty} a_k h_T(t - kT)
\]

- \( a_k \) = data symbols to be transmitted
- \( h_T(t) \) = transmitted continuous-time waveform
- \( \delta(t) \) = Dirac delta function

- \( a_k \) are assumed to be uncorrelated (white spectrum)
  \( \rightarrow \) Tx power spectrum determined by \( h_T(t) \) only!

Ideal sinc solution...

- Assume AWGN channel
- According to the water pouring theorem, the optimal Tx PSD is *constant* in the signal bandwidth \( W_0 \)
- Choose rectangular signal spectrum:

\[
H_T(f) = \frac{1}{2W_0} \text{rect}\left( \frac{f}{2W_0} \right)
= \begin{cases} 
\frac{1}{2W_0}, & |f| < W_0 \\
0, & \text{elsewhere}
\end{cases}
\]

\[
H(f) = \frac{1}{2W_0}
\]
Ideal sinc solution...

- Ideal time-domain waveform via IFT:
  \[ h_T(t) = \int_{-\infty}^{\infty} H_T(f) e^{j2\pi f t} df = \frac{1}{2W_0} \int_{-W_0}^{W_0} e^{j2\pi f t} df = \text{sinc}(2W_0 t) \]

- \( h_T(t) = 0, \ t = kT \) (except \( h_T(0) = 1 \))
  → no intersymbol interference (ISI) with any linear modulation method (PAM, QAM, etc.)
  (symbol rate \( R_S = 1/T \), ideal sampling at the receiver)

Problems:
- strictly rectangular spectrum hard to implement
  - slowly decaying, infinitely long, noncausal waveform
  - sensitivity to timing errors
  - impulse response must be truncated in digital implementation → truncation errors (Gibbs phenomenon)
II. Nyquist criterion

- Use of the flat power spectrum for the (only) design criterion leads into sinc function with:
  - zero ISI (good!)
  - implementation problems due to slow decay (bad!)
- New goal: modify the design so that
  - faster decay of time-domain response is achieved
  → use:
    » more spectrum (*excess bandwidth*)
    » less steep transition band
  - constrain zero ISI at sampling instants = *Nyquist criterion*
Nyquist criterion...

- Requirement for zero ISI:

\[ h_T(kT) = \begin{cases} 
1, & k = 0 \\
0, & k \neq 0 
\end{cases} \]

- The same with delta functions:

\[ h_t(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t-kT) = \delta(t) \]

Nyquist criterion...

- Convert into the frequency domain:

\[ h_T(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t-kT) = \delta(t) \]

\[ \Leftrightarrow \quad H_T(f) \ast \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta(f-m/T) = 1 \]

\[ \Leftrightarrow \quad \frac{1}{T} \sum_{m=-\infty}^{\infty} H_T(f-m/T) = 1 \]
Nyquist criterion...

- General form of Nyquist criterion:

  \[
  \frac{1}{T} \sum_{m=-\infty}^{\infty} H_T(f - m/T) = 1
  \]

- For a bandlimited spectrum with \( W < 2W_0 = 1/T \), we get

  \[
  H_T(f) + H_T(f - 1/T) = T
  \]

Nyquist criterion...

- The Nyquist criterion does not define a unique spectrum
  \( \rightarrow \) other constraints can be included (smooth transition!)
- Design parameter: rolloff factor \( \alpha \)
- Determines used excess bandwidth \( \alpha W_0 \)
Nyquist criterion...

- Standard choice: Raised-Cosine (RC) spectrum:

\[
H_f(f) = \begin{cases} 
  1/(2W_0), & |f| < f_i \\
  \frac{1}{4W_0} \left[1 + \cos \left(\frac{\pi |f|}{2W_0 - f_i}\right)\right] , & f_i < |f| < 2W_0 - f_i \\
  0, & |f| > 2W_0 - f_i 
\end{cases}
\]

with \( f_i = (1-\alpha)W_0 \)

Nyquist criterion...

- RC waveform via IDFT: 

\[
h(t) = \text{sinc}(2W_0 t) \frac{\cos(2\pi \alpha W_0 t)}{1 - (4\alpha^2 W_0^2 t^2)}
\]
Nyquist criterion...

Observations on RC waveform:

- Pulse decays proportionally to $1/t^3$ and $1/\alpha^2$
- Increasing $\alpha$:
  - uses more spectrum
  - improves pulse decay ($\alpha = 0$: sinc pulse)
- Typical values in practice: $\alpha = 0.1...0.5$

III. Matched filter
Matched filter

Consider receiver processing:
- If ISI is eliminated, noise is the main problem

Improve the system model:
- Add receive filter $h_R(t)$ to reduce noise

\[
\alpha_k \delta(t-kT) \quad \rightarrow \quad h_T(t) + n(t) \quad \rightarrow \quad h_R(t) \quad \rightarrow \quad \hat{a}_k \delta_k
\]

Tx waveform \hspace{2cm} Rx waveform

Matched filter...

- Known result: error probability in AWGN channel (no ISI):
  \[
P_e = Q(\sqrt{SNR})
\]

→ Rx filter design criterion: maximize SNR!
- Matched filter (MF): maximizes SNR at the sampling instant for a given Tx pulse (usually in an AWGN channel)
Matched filter...

- Design optimal Rx filter $h_R(t)$ for a given Tx filter $h_T(t)$:
- Waveform after Rx filter (without noise):
  \[ g(t) = h_T(t) * h_R(t) = \int_{-\infty}^{\infty} H_T(f) H_R(f) e^{j2\pi f t} df \]
- Symbol energy at $t = 0$:
  \[ |g(0)|^2 = \left| \int_{-\infty}^{\infty} H_T(f) H_R(f) df \right|^2 \]
- Schwarz inequality:
  \[ \left| \int_{-\infty}^{\infty} H_T(f) H_R(f) df \right|^2 \leq \int_{-\infty}^{\infty} |H_T(f)|^2 df \int_{-\infty}^{\infty} |H_R(f)|^2 df \]

Matched filter...

- Average noise power after receive filter:
  \[ E[n_R^2(t)] = \int_{-\infty}^{\infty} S_n(f) |H_R(f)|^2 df = \frac{N_0}{2} \int_{-\infty}^{\infty} |H_R(f)|^2 df \]
- Signal-to-noise ratio at sampling instant:
  \[ \text{SNR} = \frac{|g(0)|^2}{E[n_R^2(t)]} = \frac{\left| \int_{-\infty}^{\infty} H_T(f) H_R(f) df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H_R(f)|^2 df} \]
Matched filter...

- Use Schwarz inequality:

\[
\text{SNR} \leq \frac{\int |H_T(f)|^2 df \int |H_R(f)|^2 df}{N_0/2 \int |H_R(f)|^2 df} = \frac{2}{N_0} \int |H_T(f)|^2 df = \text{SNR}_{\text{MAX}}
\]

- The max SNR is obtained when

\[
H_R(f) = k_0 H_T^*(f), \quad k_0 = \text{const.}
\]

Matched filter...

- Time-domain waveform of MF:

\[
H_R(f) = k_0 H_T^*(f) \quad \Leftrightarrow \quad h_R(t) = k_0 h_T(-t)
\]

- Causal implementation:

\[
h_R(t) = k_0 h_T(t_d - t)
\]
Matched filter...

How much does MF improve over direct sampling (DS)?

1) DS:
\[ SNR_{DS} = \frac{|h_T(0)|^2}{N_0 / 2} = \frac{2}{N_0}, \quad h_T(0) = 1 \]

2) MF:
\[ SNR_{MF} = \frac{2}{N_0} \int_{-\infty}^{\infty} |H_T(f)|^2 df = \frac{2}{N_0} \int_{-\infty}^{\infty} |h_T(t)|^2 dt \]

\[ \Rightarrow \frac{SNR_{MF}}{SNR_{DS}} = \int_{-\infty}^{\infty} |h_T(t)|^2 dt = E_S \]

MF gain depends on transmitted symbol energy \( E_S \).

Explain why!

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IV. Root-Nyquist filtering
Root-Nyquist filtering

Two criteria for Tx-Rx filter optimization
◆ Nyquist criterion (to guarantee zero ISI)
◆ Matched filter (to maximize SNR at the receiver)

NOTE! MF optimal only when no ISI!

How to combine the two?

Root-Nyquist filtering...

◆ Consider a matched pulse pair:
  \[ g(t) = h_t(t)*h_R(t) = h_t(t)*h_t(-t) \]

◆ Composite frequency response:
  \[ G(f) = H_T(f)H_R^*(f) = |H_T(f)|^2 \]

→ One possible design algorithm:
1) Design \( G(f) \) as real-valued & positive Nyquist spectrum (e.g. raised-cosine)
2) Choose \( H_T(f) = H_R(f) = \sqrt{G(f)} \)
3) Get pulse waveform via IFT
Root-Nyquist filtering...

◆ Root-Raised-Cosine (RRC) spectrum:

\[
H_r(f) = \begin{cases} 
1/\sqrt{2W_0}, & |f| < f_i \\
\frac{1}{\sqrt{4W_0}} \left[1 + \cos \left(\frac{\pi(f - f_i)}{2W_0 - f_i}\right)\right], & f_i < |f| < 2W_0 - f_i \\
0, & |f| > 2W_0 - f_i 
\end{cases}
\]

with \( f_i = (1-\alpha)W_0 \)

Root-Nyquist filtering...

◆ Root-Raised-Cosine (RRC) waveform via IFT

(nice exercise for homework!):

\[
h_r(t) = \frac{\sqrt{T}}{\pi} \sin(2\pi(1-\alpha)W_0 t) \\
+ \frac{\sqrt{T}}{(2\pi)^2 - \left(\frac{\pi}{2W_0}\right)^2} \left[ (4\pi) \sin(2\pi(\alpha-1)W_0 t) \\
- \left(\frac{\pi}{2W_0}\right) \cos(2\pi(\alpha+1)W_0 t) \right]
\]
Root-Nyquist filtering...

- RRC waveform plots for different $\alpha$:

Observations on RRC waveform:
- Similar to RC
- Except: no zeros at $t = kT$!
  (Zeros come after convolution of Tx & Rx filter)
V. Tx and Rx filters in practice

◆ The RRC Tx and Rx filters are (almost) optimal (= MIN $P_e$) for AWGN channels
◆ Linear channel destroys both zero ISI and MF property
◆ However, RRC is commonly used as Tx filter and as preprocessor at the receiver
◆ In addition, for ISI elimination we usually need
  – Linear Equalizer
  – DFE Equalizer or
  – Viterbi algorithm
Tx and Rx filters in practice...

Steps for digital Tx & Rx FIR filter design:
1) Continuous-time RRC from formula
2) Sample (usually 2-4 times symbol rate)
3) Truncate symmetrically (long enough!)
4) Quantize coefficients

Summary

Today we discussed:
Transmit and receive filters for bandlimited channels
   I. Ideal sinc solution
   II. Nyquist criterion
   III. Matched filter
   IV. Root-Nyquist filtering
   V. Tx and Rx filters in practice

Next week: Optimal linear equalizers for linear channels I