S-38.411 Signal Processing in Telecommunications I
Exercise #3: Adaptive equalizers

March 10, 2000

In this exercise you will learn a little bit more about the MMSE FIR equalizers and the MSE gradient algorithm (MSEG). The MSEG algorithm is a good starting point to understand the LMS algorithm which you will get to know in more detail during the project work. Figure 1 shows a simplified block diagram of an adaptive equalizer.

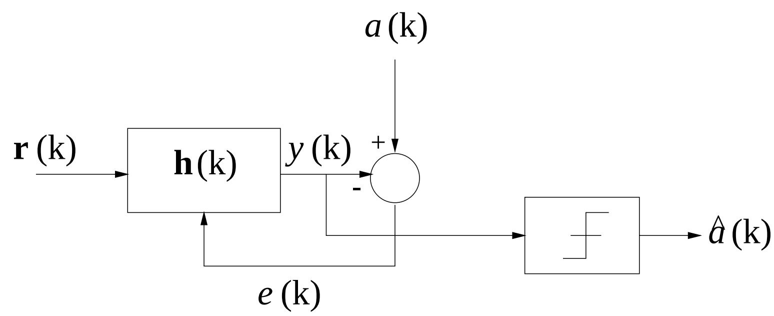

Figure 1: Block diagram over an adaptive equalizer

- $\mathbf{h}(k) = [h_{-M}(k)\ \cdots\ h_{M-1}(k)\ h_M(k)]^T$ is the adaptive receiver filter.
- $\mathbf{r}(k) = [r(k+M)\ \cdots\ r_{k-M+1}\ r(k-M)]^T$ is the input to the adaptive filter.
- $y(k)$ is the equalizer output.
- $a(k)$ is the transmitted symbol.
- $\hat{a}(k)$ is the estimate of transmitted symbol.

1. The MSE criterion

Let The MSE criterion minimizes the mean-square error of the output from the filter

$$
\begin{aligned}
MSE &= \mathrm{E}\left[e(k)^2\right] = \mathrm{E}\left[(a_k - \mathbf{h}^T\mathbf{r}(k))^2\right] \\
&= \mathrm{E}\left[a(k)^2\right] - 2\mathbf{h}^T\mathbf{p} + \mathbf{h}^T\mathbf{R}\mathbf{h}
\end{aligned}
\tag{1}
$$

where $\mathbf{R}$ and $\mathbf{p}$ given by

$$
\begin{aligned}
\mathbf{R} &= \mathrm{E}\left[\mathbf{r}(k)\mathbf{r}^T(k)\right] \\
\mathbf{p} &= \mathrm{E}\left[\mathbf{r}(k)a(k)\right]
\end{aligned}
\tag{2}
$$

are the autocorrelation matrix and the crosscorrelation vector respectively.
The optimum filter coefficients $\mathbf{h}$ that minimize the MSE are given by

$$
\mathbf{h}_{opt} = \mathbf{R}^{-1}\mathbf{p} \tag{3}
$$

Substituting (3) into (1) give us the minimum MSE (MSE_{min})

$$
\begin{aligned}
MSE_{min} &= \mathrm{E}\left[a(k)^2\right] - \mathbf{p}^T\mathbf{R}^{-1}\mathbf{p} \\
&= \mathrm{E}\left[a(k)^2\right] - \mathbf{h}_{opt}^T\mathbf{p}
\end{aligned}
\tag{4}
$$

1. The MSEG algorithm

The optimal MSE solution for the receive filter $\mathbf{h}_{opt}$ above requires knowledge of the autocorrelation $\mathbf{R}$ and crosscorrelation vector $\mathbf{p}$. In practice these need to be estimated due to the unknown channel.

The MSEG algorithm is an iterative algorithm searching for the solution to the equation system $\mathbf{Rh} = \mathbf{p}$. The update formula is given by (see Lecture 6)

$$
\mathbf{h}^{(j+1)} = \mathbf{h}^{(j)} - \beta\left(\mathbf{p} - \mathbf{R}\mathbf{h}^{(j)}\right) \tag{5}
$$

where β is the step size satisfying

$$
0 < \beta < \frac{2}{\lambda_{max}} \tag{6}
$$

where λ_{max} is the maximum eigenvalue of $\mathbf{R}$. Note that j is the iteration index (sample index). If $\mathbf{R}$ and $\mathbf{p}$ unknown they need to be estimated (see Lecture 7, page 7).

Exercise 1:

We start with some easy stuff! Which of the matrix below is a valid autocorrelation matrix. Motivate your answers.

$$\text{a) } \mathbf{R}_1 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 0 & 4 \end{bmatrix} \quad \text{b) } \mathbf{R}_1 = \begin{bmatrix} 1 & -0.5 & 0.2 \\ -0.5 & 1 & -0.5 \\ 0.2 & -0.5 & 1 \end{bmatrix} \quad \text{c) } \mathbf{R}_1 = \begin{bmatrix} 1 & -0.5 & 0.2 \\ -0.5 & -0.5 & -0.5 \\ 0.2 & -0.5 & 1 \end{bmatrix} \tag{7}$$

Exercise 2:

Suppose the autocorrelation matrix $\mathbf{R}$ and the crosscorrelation vector $\mathbf{p}$ are known for a given experimental environment.

$$\mathbf{R} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \quad \mathbf{p} = [0.5\; 0.25]^T \tag{8}$$

a) Calculate the optimum coefficient vector $\mathbf{h}_{opt}$ (Wiener solution)

b) What is the minimum MSE assuming that the variance of the transmitted signal $\sigma_a^2 = \mathrm{E}\left[a_k^2\right] = 1$?

c) State the update formula for the MSEG algorithm.

d) What is the maximum step-size that can be used $(2/\lambda_{max})$?

e) Consider that the adaptive filter coefficients are initially zero. Calculate their values for the first 3 iterations using the maximum step size from d)?

Extra exercise to practice at home:

Repeat Exercice 2 using $\mathbf{R}$ and $\mathbf{p}$ below, and the variance of the transmitted signal $\sigma_a^2 = 5$

$$\mathbf{R} = \frac{1}{3} \begin{bmatrix} 3 & -2 & 1 \\ -2 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix} \quad \mathbf{p} = \left[-2\; 1\; -\frac{1}{2}\right]^T \tag{9}$$

Homework 3

The homework is to be returned to the course box *at latest* March 24, 15:00. The course box can be found near the course information board on the second floor in the G wing. Each set of homework can give up to 1 point on the final exam. Remember to motivate each step in your solution. Write your name and student number on each page.

1. Let the output from a linear channel be given by $r(k) = c_1 a(k-1) + c_2 a(k-2) + c_3 a(k-3) + n(k)$, where the c_i:s are channel coefficients, $a(k) \in \pm 1$ is the transmitted signal sequence, and $n(k)$ is additive white noise with variance σ_n^2.

   In this homework you are to design and implement an adaptive equalizer of length $L = 3$ using the MSEG algorithm. The structures of the equalizer filter $\mathbf{h}(k)$ and the input vector $\mathbf{r}(k)$ are given by:
   $\mathbf{h}(k) = [h_{-1}(k)\; h_0(k)\; h_1(k)]^T$ and $\mathbf{r}(k) = [r(k+1)\; r(k)\; r(k-1)]^T$

   a) Calculate the autocorrelation matrix $\mathbf{R} = \mathrm{E}\left[\mathbf{r}(k)\mathbf{r}^T(k)\right]$ and the crosscorrelation vector $\mathbf{p} = \mathrm{E}\left[\mathbf{r}(k)a(k)\right]$

   b) Calculate the optimum coefficient vector (Wiener solution) assuming $c_1 = 0.3$, $c_2 = 0.8$, $c_3 = 0.3$ and $\sigma_n^2 = 0.001$

   c) Calculate the minimum MSE.

   d) Calculate the maximum step size that can be used in the MSEG to guarantee stability.

   e) Consider that the adaptive filter coefficients are initially zero. What are their values after 10 iterations using the MSEG algorithm.