S-38.3143
Queueing Theory

Basic probability theory

Samuli Aalto
TKK Helsinki University of Technology

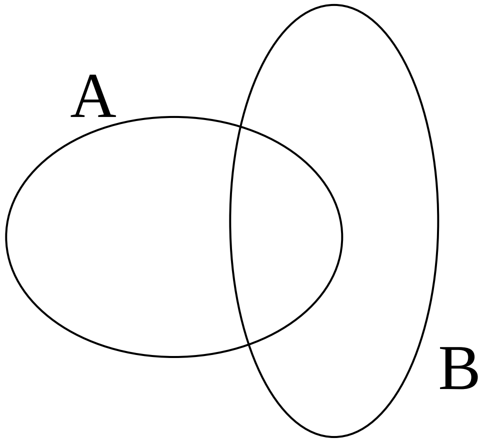
Contents

Sample space, sample points, events

- **Sample space Ω** is the set of all possible **sample points $\omega \in \Omega$**
- **Events $A,B,C,... \subset \Omega$** are measurable subsets of the sample space Ω
- Let $\mathcal{F}$ denote the set of all events, which constitutes a σ-algebra
  - **Sure event**: The sample space $\Omega \in \mathcal{F}$
  - **Impossible event**: The empty set $\varnothing \in \mathcal{F}$
  - **Union** "A or B": $A \cup B = \{\omega \in \Omega \mid \omega \in A \text{ or } \omega \in B\} \in \mathcal{F}$
  - **Intersection** "A and B": $A \cap B = \{\omega \in \Omega \mid \omega \in A \text{ and } \omega \in B\} \in \mathcal{F}$
  - **Complement** "not A": $A^c = \{\omega \in \Omega \mid \omega \notin A\} \in \mathcal{F}$
  - Events A and B are **disjoint** if $A \cap B = \varnothing$
  - A set of events $\{B_1, B_2, \ldots\}$ is a **partition** of event A if
    - (i) $B_i \cap B_j = \varnothing$ for all $i \neq j$
    - (ii) $\cup_i B_i = A$

Probability

- **Probability** of event A is denoted by $P(A)$, $P(A) \in [0,1]$
  - Probability measure P is thus
    a real-valued set function defined on the set $\mathcal{F}$ of events, $P: \mathcal{F} \to [0,1]$
- **Properties**:
  - (i) $0 \le P(A) \le 1$
  - (ii) $P(\varnothing) = 0$
  - (iii) $P(\Omega) = 1$
  - (iv) $P(A^c) = 1 - P(A)$
  - (v) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - (vi) $A \cap B = \varnothing \Rightarrow P(A \cup B) = P(A) + P(B)$
  - (vii) $\{B_i\}$ is a partition of $A \Rightarrow P(A) = \Sigma_i P(B_i)$
  - $(viii)$ $A \subset B \Rightarrow P(A) \le P(B)$



Conditional probability

- Assume that $P(B) > 0$
- **Definition**: The **conditional probability** of event A **given** that event B occurred is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- It follows that

$$P(A \cap B) = P(B)P(A \mid B) = P(A)P(B \mid A)$$

Theorem of total probability

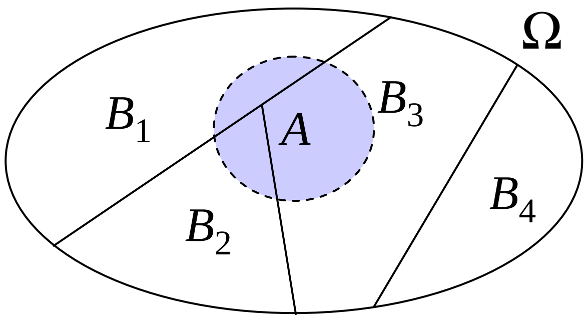
- Let $\{B_i\}$ be a partition of the sample space Ω
- It follows that $\{A \cap B_i\}$ is a partition of event A. Thus (by slide 5)

$$P(A) \overset{(vii)}{=} \sum_i P(A \cap B_i)$$

- Assume further that $P(B_i) > 0$ for all i. Then (by slide 6)

$$P(A) = \sum_i P(B_i) P(A \mid B_i)$$

- This is the **theorem of total probability**



Statistical independence of events

- **Definition**: Events A and B are **independent** if

$$P(A \cap B) = P(A)P(B)$$

- It follows that

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

- Correspondingly:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B)$$

Random variables

- **Definition**: Real-valued **random variable** X is a real-valued and measurable function defined on the sample space Ω, X: $\Omega \rightarrow \Re$
  - Each sample point $\omega \in \Omega$ is associated with a real number $X(\omega)$
- **Measurability** means that all sets of type

$$\{X \leq x\} := \{\omega \in \Omega \mid X(\omega) \leq x\} \subset \Omega$$

belong to the set $\mathcal{F}$ of events, i.e.,

$$\{X \leq x\} \in \mathcal{F}$$

- The probability of such an event is denoted by $P\{X \leq x\}$
- **Notation**: Capital Letters (such as X) refer to random variables, while small letters (such as x) refer to their values

Indicators of events

- Let $A \in \mathcal{F}$ be an arbitrary event
- **Definition**: The **indicator** of event A is a random variable defined by

$$1_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

- Clearly:

$$P\{1_A = 1\} = P(A)$$

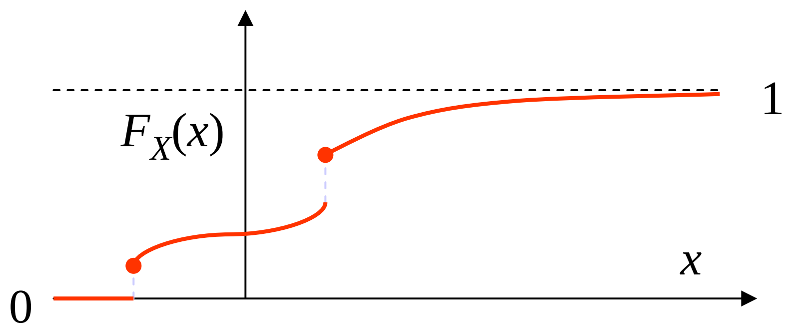
$$P\{1_A = 0\} = P(A^c) = 1 - P(A)$$

Cumulative distribution function

- **Definition**: The **cumulative distribution function** (CDF) of a random variable X is a function $F_X\colon \Re \to [0,1]$ defined as follows:

$$F_X(x) := P\{X \le x\}$$

- Cdf determines the **distribution** of the random variable, i.e.,
  - the probabilities $P\{X \in B\}$, where $B \subset \Re$ and $\{X \in B\} \in \mathcal{F}$
- **Properties**:
  - (i) F_X is non-decreasing
  - (ii) F_X is continuous from the right
  - (iii) $F_X(-\infty) = 0$
  - (iv) $F_X(\infty) = 1$



Statistical independence of random variables

- **Definition**: Random variables X and Y are **independent** if for all x and y

$$P\{X \le x, Y \le y\} = P\{X \le x\}P\{Y \le y\}$$

- **Definition**: Random variables $X_1,\ldots, X_n$ are **totally independent** if for all i and x_i

$$P\{X_1 \le x_1,\ldots, X_n \le x_n\} = P\{X_1 \le x_1\}\cdots P\{X_n \le x_n\}$$

- **Definition**: Random variables $X_1,\ldots, X_n$ are **IID** if they are totally independent and identically distributed

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- Basic concepts
- Discrete random variables
- Conditional expectation and variance
- Probability generating function (z-transform)
- Discrete distributions (count distributions)
- Continuous random variables
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- Continuous distributions (time distributions)
- Other distributions and random variables

Discrete random variables

- **Definition**: Set $A \subset \Re$ is called **discrete** if it is
  - finite, $A = \{x_1, \ldots, x_n\}$, or
  - countably infinite, $A = \{x_1, x_2, \ldots\}$
- **Definition**: Random variable X is **discrete** if there is a discrete set $S_X \subset \Re$ such that

$$P\{X \in S_X\} = 1$$

- It follows that
  - $P\{X = x\} \geq 0$ for all $x \in S_X$
  - $P\{X = x\} = 0$ for all $x \notin S_X$
- The set S_X is called the **value space**

Point probabilities

- Let X be a discrete random variable
- The distribution of X is determined by the **point probabilities** p_i,

$$p_i := P\{X = x_i\}, \quad x_i \in S_X$$

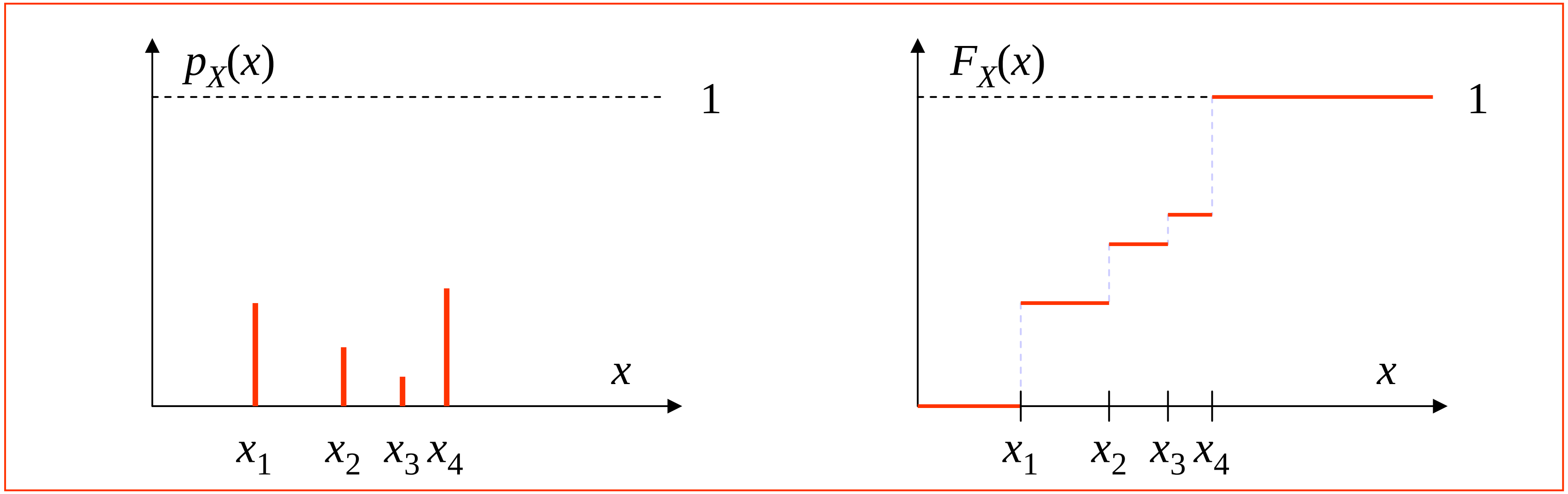
- **Definition**: The **probability mass function** (PMF) of X is a function $p_X\colon \Re \to [0,1]$ defined as follows:

$$p_X(x) := P\{X = x\} = \begin{cases} p_i, & x = x_i \in S_X \\ 0, & x \notin S_X \end{cases}$$

- CDF is in this case a step function:

$$F_X(x) = P\{X \leq x\} = \sum_{i:x_i \leq x} p_i$$

Example



probability mass function (PMF) cumulative distribution function (CDF)

$$S_X = \{x_1, x_2, x_3, x_4\}$$

Expectation

- **Definition**: The **expectation** (mean value) of a discrete random variable X is defined by

$$E[X] := \sum_{x \in S_X} P\{X = x\} \cdot x$$

  - Note 1: The expectation exists only if $\Sigma_x |x|\, P\{X = x\} < \infty$
  - Note 2: However, if $\Sigma_x\, x\, P\{X = x\} = \infty$, then we may denote $E[X] = \infty$
  - Note 3: Expectation of an indicator: $E[1_A] = P\{1_A = 1\} = P(A)$
- **Properties**:
  - $(i)\quad c \in \Re \Rightarrow E[cX] = cE[X]$
  - $(ii)\quad E[X + Y] = E[X] + E[Y]$
  - (iii) X and Y independent $\Rightarrow E[XY] = E[X]E[Y]$

Variance

- **Definition**: The **variance** of X is defined by

$$D^2[X] := \mathrm{Var}[X] := E[(X - E[X])^2]$$

- Useful formula:

$$D^2[X] = E[X^2] - E[X]^2$$

- **Properties**:
  - (i) $c \in \Re \Rightarrow D^2[cX] = c^2 D^2[X]$
  - (ii) $D^2[X+Y] = D^2[X] + D^2[Y] + 2\,\mathrm{Cov}\,[X,Y]$ (see next slide)
  - (iii) X and Y independent $\Rightarrow D^2[X+Y] = D^2[X] + D^2[Y]$

Covariance

- **Definition**: The **covariance** between X and Y is defined by

$$\mathrm{Cov}[X,Y] := E[(X - E[X])(Y - E[Y])]$$

- Useful formula:

$$\mathrm{Cov}[X,Y] = E[XY] - E[X]E[Y]$$

- **Properties**:
  - (i) $\mathrm{Cov}[X,X] = \mathrm{Var}[X]$
  - (ii) $\mathrm{Cov}[X,Y] = \mathrm{Cov}[Y,X]$
  - (iii) $\mathrm{Cov}[X+Y,Z] = \mathrm{Cov}[X,Z] + \mathrm{Cov}[Y,Z]$
  - (iv) X and Y independent $\Rightarrow \mathrm{Cov}[X,Y] = 0$

Other distribution related parameters

- **Definition**: The **standard deviation** of X is defined by

$$D[X] := \sqrt{D^2[X]}$$

- **Definition**: The **coefficient of variation** of X is defined by

$$C[X] := \frac{D[X]}{E[X]}$$

- **Definition**: The *k*th **moment**, $k=1,2,\ldots$, of X is defined as

$$E[X^k] = \sum_x P\{X = x\} \cdot x^k$$

Average of IID random variables

- Let $X_1, \ldots, X_n$ be independent and identically distributed (**IID**) with mean μ and variance σ^2
- Denote the average (sample mean) as follows:

$$\overline{X}_n := \frac{1}{n} \sum_{i=1}^{n} X_i$$

- Then

$$E[\overline{X}_n] = \mu$$

$$D^2[\overline{X}_n] = \frac{\sigma^2}{n}$$

$$D[\overline{X}_n] = \frac{\sigma}{\sqrt{n}}$$

Law of large numbers (LLN)

- Let $X_1,\ldots, X_n$ be independent and identically distributed (**IID**) with mean μ and variance σ^2
- **Weak law of large numbers**: for all $\varepsilon > 0$

$$P\{|\bar{X}_n - \mu| > \varepsilon\} \to 0$$

- **Strong law of large numbers**: with probability 1

$$\bar{X}_n \to \mu$$

- It follows that for large values of n

$$\bar{X}_n \approx \mu$$

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Conditional expectation

- **Definition**: Let X and Y be discrete random variables. The **conditional expectation** of X (conditioned on Y) is a random variable defined by the following function (taking values in S_Y):

$$E[X \mid Y = y] := \sum_{x \in S_X} P\{X = x \mid Y = y\} \cdot x$$

$$E[X \mid Y] := f(Y), \text{ where } f(y) := E[X \mid Y = y]$$

- **Properties**:
  - (i) $E[g(Y)\, X|Y] = g(Y)\, E[X|Y]$
  - (ii) $E[X + Y|Z] = E[X|Z] + E[Y|Z]$
  - (iii) X and Y independent $\Rightarrow E[X|Y] = E[X]$
  - (iv) $E[E[X|Y]] = E[X]$ (conditioning rule)

Conditional variance

- **Definition**: Let X and Y be discrete random variables. The **conditional variance** of X (conditioned on Y) is a function defined on the value space S_Y of Y by

$$D^2[X \mid Y] := E[(X - E[X \mid Y])^2 \mid Y]$$

- Useful formula:

$$D^2[X] = E[D^2[X \mid Y]] + D^2[E[X \mid Y]]$$

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Probability generating function (z-transform)

- **Definition**: Let X be a discrete random variable taking values in $S_X = \{0,1,2,\ldots\}$. The **probability generating function** (PGF) of X is defined by

$$G_X(z) := E[z^X] = \sum_{i=0}^{\infty} P\{X = i\} z^i, \quad |z| \le 1$$

- PGF is also known as the z-transform of X
- PGF determines the distribution unambiguously:

$$\frac{d^k}{dz^k} G_X(0) = k! P\{X = k\}$$

Factorial moments

- PGF generates the **factorial moments** of X :

$$G_X(1) = 1$$

$$\frac{d}{dz} G_X(1) = E[X]$$

$$\frac{d^2}{dz^2} G_X(1) = E[X(X-1)]$$

$$\frac{d^k}{dz^k} G_X(1) = E[X(X-1)\ldots(X-k+1)]$$

Sum of independent random variables

- Let X and Y be independent random variables taking values in $\{0,1,2,\ldots\}$. The PGF of the sum $X+Y$ is given by

$$G_{X+Y}(z) = E[z^{X+Y}] = E[z^X]E[z^Y] = G_X(z)G_Y(z)$$

- Let $X_1,\ldots,X_n$ be IID random variables taking values in $\{0,1,2,\ldots\}$. The PGF of the sum $Y = X_1 + \ldots + X_n$ is given by

$$\begin{aligned} G_Y(z) &= E[z^Y] \\ &= E[z^{X_1+\ldots+X_n}] \\ &= E[z^{X_1}]\ldots E[z^{X_n}] = G_X(z)^n \end{aligned}$$

Random sum of independent random variables

- Let $X_1, X_2, \ldots$ be IID random variables taking values in $\{0,1,2,\ldots\}$. In addition, let N be another independent random variable taking values in $\{0,1,2,\ldots\}$. The PGF of the random sum $Y = X_1 + \ldots + X_N$ is given by

$$
\begin{aligned}
G_Y(z) &= E[z^Y] = E[E[z^Y \mid N]] \\
&= E[E[z^{X_1+\ldots+X_N} \mid N]] \\
&= E[E[z^{X_1}]\ldots E[z^{X_N}]] \\
&= E[G_X(z)^N] = G_N(G_X(z))
\end{aligned}
$$

Method of collective marks

- Probabilistic interpretation of the z-transform $G_X(z) = E[z^X]$:
  - Think of X as representing the size of some (random) set.
  - Mark each of the elements in the set independently with probability $1 - z$ and leave it unmarked with probability z.
  - Then $G_X(z)$ is the *probability that there is no mark in the whole set*.

$$G_X(z) = E[z^X] =$$
$$= E[P\{\text{Set of size } X \text{ has no mark}\} \mid X]$$
$$= E[E[1_{\{\text{Set of size } X \text{ has no mark}\}} \mid X]]$$
$$= E[1_{\{\text{Set of size } X \text{ has no mark}\}}]$$
$$= P\{\text{Set of size } X \text{ has no mark}\}$$

Contents

Bernoulli distribution

$$X \sim \text{Bernoulli}(p), \quad p \in (0,1)$$

- describes a simple random experiment (called **Bernoulli trial**) with two possible outcomes: success (1) and failure (0); cf. coin tossing
- success with probability p (and failure with probability $1 - p$)

- Value space: $S_X = \{0,1\}$
- Point probabilities:

$$P\{X = 0\} = 1 - p, \quad P\{X = 1\} = p$$

- Mean value: $E[X] = (1 - p)\cdot 0 + p\cdot 1 = p$
- Second moment: $E[X^2] = (1 - p)\cdot 0^2 + p\cdot 1^2 = p$
- Variance: $D^2[X] = E[X^2] - E[X]^2 = p - p^2 = p(1 - p)$
- PGF: $E[z^X] = 1 - p + pz$

Binomial distribution

$$X \sim \mathrm{Bin}(n, p), \quad n \in \{1,2,...\}, p \in (0,1)$$

- number of successes in a finite sequence of IID Bernoulli trials; $X = X_1 + \ldots + X_n$ with $X_i \sim \mathrm{Bernoulli}(p)$
- n = total number of experiments
- p = probability of success in any single experiment

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

$$n! = n \cdot (n-1) \cdots 2 \cdot 1$$

- Value space: $S_X = \{0,1,\ldots,n\}$
- Point probabilities:

$$P\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i}$$

- Mean value: $E[X] = E[X_1] + \ldots + E[X_n] = np$
- Variance: $D^2[X] = D^2[X_1] + \ldots + D^2[X_n] = np(1-p)$
- PGF: $E[z^X] = E[z^{X_1}] \cdot \ldots \cdot E[z^{X_n}] = (1 - p + pz)^n$

Sum property

- Let $X = X_1 + \ldots + X_n$ where $X_i \sim \text{Bin}(n_i,p)$ and are independent. Then

$$X \sim \text{Bin}(\textstyle\sum_i n_i, p)$$

Geometric distribution

$$X \sim \mathrm{Geom}(p), \quad p \in (0,1)$$

- number of successes until the first failure in a sequence of IID Bernoulli trials
- p = probability of success in any single experiment

- Value space: $S_X = \{0,1,\ldots\}$
- Point probabilities:

$$P\{X = i\} = p^i (1-p)$$

- Mean value: $E[X] = \sum_i ip^i(1-p) = p/(1-p)$
- Second moment: $E[X^2] = \sum_i i^2p^i(1-p) = 2(p/(1-p))^2 + p/(1-p)$
- Variance: $D^2[X] = E[X^2] - E[X]^2 = p/(1-p)^2$
- PGF: $E[z^X] = \sum_i (pz)^i(1-p) = (1-p)/(1-pz)$

Memoryless property

- Geometric distribution has so called **memoryless property**:
  for all $i,j \in \{0,1,...\}$ so

$$P\{X \geq i+j \mid X \geq i\} = P\{X \geq j\}$$

  - Note: $P\{X \geq i\} = p^i$

Minimum of geometric random variables

- Let $X_1 \sim \text{Geom}(p_1)$ and $X_2 \sim \text{Geom}(p_2)$ be **independent**. Then

$$X^{\min} := \min\{X_1, X_2\} \sim \text{Geom}(p_1 p_2)$$

and

$$P\{X^{\min} = X_i\} = \frac{1 - p_i}{1 - p_1 p_2}, \quad i \in \{1,2\}$$

Poisson distribution

$$X \sim \text{Poisson}(a), \quad a > 0$$

– limit of binomial distribution as $n \to \infty$ and $p \to 0$ in such a way that $np \to a$

- Value space: $S_X = \{0,1,\ldots\}$
- Point probabilities:

$$P\{X = i\} = \frac{a^i}{i!} e^{-a}$$

- Mean value: $E[X] = a$
- Second moment: $E[X(X-1)] = a^2 \Rightarrow E[X^2] = a^2 + a$
- Variance: $D^2[X] = E[X^2] - E[X]^2 = a$
- PGF: $E[z^X] = \sum_i (az)^i e^{-a}/i! = e^{-a} e^{az} = e^{-a(1-z)}$

Properties

- (i) **Sum**: Let $X_1 \sim \text{Poisson}(a_1)$ and $X_2 \sim \text{Poisson}(a_2)$ be independent. Then

$$X_1 + X_2 \sim \text{Poisson}(a_1 + a_2)$$

- (ii) **Random sample**: Let $X \sim \text{Poisson}(a)$ denote the number of elements in a set, and Y denote the size of a random sample of this set (each element taken independently with probability p). Then

$$Y \sim \text{Poisson}(pa)$$

- (iii) **Random sorting**: Let X and Y be as in (ii), and $Z = X - Y$. Then Y and Z are **independent** (given that X is unknown) and

$$Z \sim \text{Poisson}((1-p)a)$$

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- Continuous distributions (time distributions)
- Other distributions and random variables

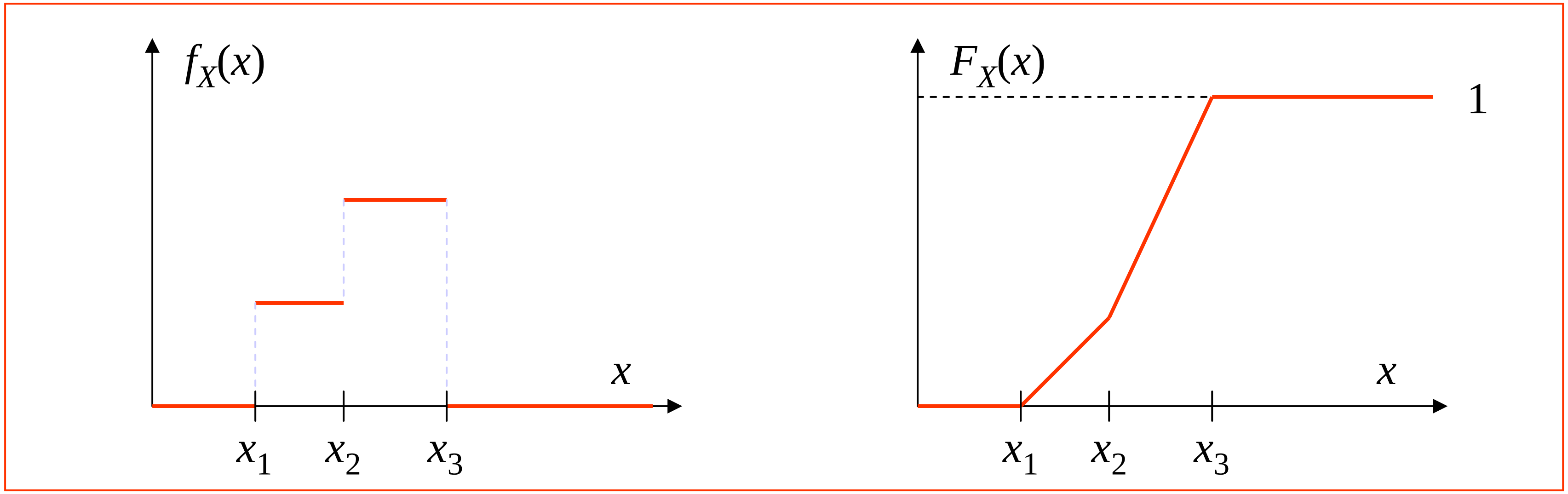
Continuous random variables

- **Definition**: Random variable X is **continuous** if there is an integrable function f_X: $\Re \rightarrow \Re_+$ such that for all $x \in \Re$

$$F_X(x) := P\{X \leq x\} = \int_{-\infty}^{x} f_X(y)\, dy$$

  - The function f_X is called the **probability density function** (PDF)
  - The set S_X, where $f_X > 0$, is called the **value space**
- **Properties**:
  - (i) $P\{X = x\} = 0$ for all $x \in \Re$
  - (ii) $P\{a < X < b\} = P\{a \leq X \leq b\} = \int_a^b f_X(x)\, dx$
  - (iii) $P\{X \in A\} = \int_A f_X(x)\, dx$
  - (iv) $P\{X \in \Re\} = \int_{-\infty}^{\infty} f_X(x)\, dx = \int_{S_X} f_X(x)\, dx = 1$

Example



probability density function (PDF) cumulative distribution function (CDF)

$$S_X = [x_1, x_3]$$

Expectation and other distribution related parameters

- **Definition**: The **expectation** (mean value) of X is defined by

$$E[X] := \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

  - Note 1: The expectation exists only if $\int_{-\infty}^{\infty} f_X(x)|x|\, dx < \infty$
  - Note 2: If $\int_{-\infty}^{\infty} f_X(x)x = \infty$, then we may denote $E[X] = \infty$
  - The expectation has the same properties as in the discrete case
- The other distribution parameters (variance, covariance,...) are defined just as in the discrete case
  - These parameters have the same properties as in the discrete case

Contents

Laplace transform

- **Definition**: Let X be a continuous random variable taking values in $S_X = (0, \infty)$. The **Laplace transform** (LT) of X is defined by

$$L_X(s) := E[e^{-sX}] = \int_0^\infty e^{-sx} f_X(x)\, dx, \quad s \geq 0$$

- LT determines the distribution unambiguously

Moments

- LT generates the **moments** of X:

$$L_X(0) = 1$$

$$\frac{d}{ds} L_X(0) = -E[X]$$

$$\frac{d^2}{ds^2} L_X(0) = E[X^2]$$

$$\frac{d^k}{ds^k} L_X(0) = (-1)^k E[X^k]$$

Sum of independent random variables

- Let X and Y be independent positive random variables. The LT of the sum $X+Y$ is given by

$$L_{X+Y}(s) = E[e^{-s(X+Y)}] = E[e^{-sX}]E[e^{-sY}] = L_X(s)L_Y(s)$$

- Let $X_1,\ldots,X_n$ be IID positive random variables. The LT of the sum $Y = X_1 + \ldots + X_n$ is given by

$$\begin{aligned} L_Y(s) &= E[e^{-sY}] \\ &= E[e^{-s(X_1+\ldots+X_n)}] \\ &= E[e^{-sX_1}]\ldots E[e^{-sX_n}] = L_X(s)^n \end{aligned}$$

Random sum of independent random variables

- Let $X_1, X_2, \ldots$ be IID positive random variables. In addition, let N be another independent random variable taking values in $\{0,1,2,\ldots\}$. The LT of the random sum $Y = X_1 + \ldots + X_N$ is given by

$$
\begin{aligned}
L_Y(s) = E[e^{-sY}] &= E[E[e^{-sY} \mid N]] \\
&= E[E[e^{-s(X_1+\ldots+X_N)} \mid N]] \\
&= E[E[e^{-sX_1}]\ldots E[e^{-sX_N}]] \\
&= E[L_X(s)^N] = G_N(L_X(s))
\end{aligned}
$$

Method of collective marks

- Probabilistic interpretation of the Laplace transform $L_X(s) = E[e^{-sX}]$:
  - Think of X as representing the length of an interval.
  - Let this interval be subject to an independent Poisson marking process with intensity s.
  - Then $L_X(s)$ is the *probability that there is no mark in the whole interval*.

$$
\begin{aligned}
L_X(s) &= E[e^{-sX}] = \\
&= E[P\{\text{Interval}[0, X]\,\text{has no mark}\} \mid X] \\
&= E[E[1_{\{\text{Interval}[0,X]\,\text{has no mark}\}} \mid X]] \\
&= E[1_{\{\text{Interval}[0,X]\,\text{has no mark}\}}] \\
&= P\{\text{Interval}[0, X]\,\text{has no mark}\}
\end{aligned}
$$

Catastrophe process

- Another probabilistic interpretation of the Laplace transform $L_X(s)$:
  - Measuring from time 0, let X represent the time of an event.
  - Independent of that, a catastrophe happens at time C, where C is exponentially distributed with intensity s.
  - Then $L_X(s)$ is the *probability that the event occurs before catastrophe*.

$$
\begin{aligned}
L_X(s) = E[e^{-sX}] &= \\
&= E[P\{C > X\} \mid X] \\
&= E[E[1_{\{C>X\}} \mid X]] \\
&= E[1_{\{C>X\}}] \\
&= P\{C > X\}
\end{aligned}
$$

Contents

From geometric to exponential distribution

- Assume that $X_n \sim \mathrm{Geom}(1 - \lambda/n)$ for some $\lambda > 0$. Now

$$P\{X_n > nx\} = (1 - \frac{\lambda}{n})^{nx} \rightarrow e^{-\lambda x}$$

- Thus, the asymptotic CDF of the rescaled random variable X_n/n is

$$F(x) = 1 - e^{-\lambda x}$$

Exponential distribution

$$X \sim \mathrm{Exp}(\lambda), \quad \lambda > 0$$

- continuous counterpart of the geometric distribution ("failure" prob. $\approx \lambda dt$)
- $P\{X \in (t,t+h] \mid X > t\} = \lambda h + o(h)$, where $o(h)/h \to 0$ as $h \to 0$

- Value space: $S_X = (0,\infty)$
- PDF and CDF:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0$$

$$F_X(x) := P\{X \le x\} = 1 - e^{-\lambda x}$$

Moments and the Laplace transform

$$X \sim \mathrm{Exp}(\lambda), \quad \lambda > 0$$

- Mean value: $E[X] = \int_0^\infty \lambda x\, e^{-\lambda x}\, dx = 1/\lambda$
- Second moment: $E[X^2] = \int_0^\infty \lambda x^2\, e^{-\lambda x}\, dx = 2/\lambda^2$
- Variance: $D^2[X] = E[X^2] - E[X]^2 = 1/\lambda^2$
- Standard deviation: $D[X] = \sqrt{D^2[X]} = 1/\lambda$
- Coefficient of variation: $C[X] = D[X]/E[X] = 1$
- Laplace transform: $E[e^{-sX}] = \int_0^\infty \lambda\, e^{-(\lambda+s)x}\, dx = \lambda/(\lambda+s)$

Memoryless property and the residual lifetime

- Exponential distribution has so called **memoryless property**:
  for all $x,y \in (0,\infty)$

$$P\{X > x + y \mid X > x\} = P\{X > y\}$$

  - Note: $P\{X > x\} = e^{-\lambda x}$
- In fact, only the exponential distribution has this property (among the continuous distributions)
- Consider a random interval whose length $X \sim \mathrm{Exp}(\lambda)$. Assume that we know that the interval is longer than x. Due to the memoryless property, the **residual lifetime** is also exponentially distributed with mean $1/\lambda$:

$$\mathrm{MRL}(x) := E[X - x \mid X > x] = \frac{1}{\lambda}$$

- Thus, the mean residual lifetime function $\mathrm{MRL}(x)$ is constant

Hazard rate

- Consider a random interval whose length $X \sim \mathrm{Exp}(\lambda)$. Assume that we know that the interval is longer than x. What is the probability that it will end in an infinitesimal interval of length h after time x?

$$P\{X \le x+h \mid X > x\} = P\{X \le h\} = 1-e^{-\lambda h}$$

$$= 1-(1-\lambda h+\tfrac{1}{2}(\lambda h)^2-\ldots) = \lambda h + o(h)$$

- Thus, in the limit ($h \to 0$), the ending probability per time unit (**hazard rate**) is constant:

$$h(x) := \lim_{h\to 0} \tfrac{1}{h} P\{X \le x+h \mid X > x\} = \lambda$$

- Again, only the exponential distribution has this property

Minimum of exponential random variables

- Let $X_1,\ldots, X_n$ be **independent** random variables with $X_i \sim \text{Exp}(\lambda_i)$. Then

$$X^{\min} := \min\{X_1,\ldots,X_n\} \sim \text{Exp}(\lambda_1+\ldots+\lambda_n), \text{since}$$

$$P\{X^{\min} > x\} = P\{X_1 > x\}\ldots P\{X_n > x\} = e^{-(\lambda_1+\ldots+\lambda_n)x}$$

$$E[X^{\min}] = \frac{1}{\lambda_1+\ldots+\lambda_n}, \qquad P\{X^{\min} = X_i\} = \frac{\lambda_i}{\lambda_1+\ldots+\lambda_n}$$

Maximum of exponential random variables

- Let $X_1,\ldots,X_n$ be **IID** random variables with $X_i \sim \mathrm{Exp}(\lambda)$. Then

$$X^{\max} := \max\{X_1,\ldots,X_n\}$$

$$P\{X^{\max} \le x\} = P\{X_1 \le x\}\ldots P\{X_n \le x\} = (1-e^{-\lambda x})^n$$

$$E[X^{\max}] = \frac{1}{n\lambda} + \frac{1}{(n-1)\lambda} + \ldots + \frac{1}{\lambda}, \qquad P\{X^{\max} = X_i\} = \frac{1}{n}$$

Erlang distribution

$$X \sim \mathrm{Erl}(n,\mu), \quad \mu > 0$$

- IID exponential phases in a series; $X = X_1 + \ldots + X_n$ where $X_i \sim \mathrm{Exp}(\mu)$
- n = total number of phases
- μ = intensity of any single phase

- Value space: $S_X = (0,\infty)$
- PDF and CDF:

$$f_X(x) = \mu \frac{(\mu x)^{n-1}}{(n-1)!} e^{-\mu x}, \qquad x > 0$$

$$F_X(x) := P\{X \le x\} = 1 - \sum_{i=0}^{n-1} \frac{(\mu x)^i}{i!} e^{-\mu x}$$

Moments and the Laplace transform

$$X \sim \mathrm{Erl}(n, \mu), \quad \mu > 0$$

- Mean value: $E[X] = E[X_1] + \ldots + E[X_n] = n/\mu$
- Variance: $D^2[X] = D^2[X_1] + \ldots + D^2[X_n] = n/\mu^2$
- Second moment: $E[X^2] = E[X]^{\,2} + D^2[X] = n(n+1)/\mu^2$
- Standard deviation: $D[X] = \sqrt{D^2[X]} = (\sqrt{n})/\mu$
- Coefficient of variation: $C[X] = D[X]/E[X] = 1/(\sqrt{n}) \leq 1$
- Laplace transform : $E[e^{-sX}] = E[e^{-sX_1}] \cdot \ldots \cdot E[e^{-sX_n}] = (\mu/(\mu+s))^n$

Mean residual lifetime

- Consider a random interval whose length $X \sim \mathrm{Erl}(n,\mu)$.
  Assume that we know that the interval is longer than x.
  What is the mean residual lifetime?

$$\mathrm{MRL}(x) := E[X - x \mid X > x]$$

$$= \frac{\int_x^\infty (1 - F_X(y))\,dy}{1 - F_X(x)} = \frac{1}{\mu} \cdot \frac{\sum_{i=0}^{n-1} (n-i)\frac{(\mu x)^i}{i!}}{\sum_{i=0}^{n-1} \frac{(\mu x)^i}{i!}}$$

- The mean residual lifetime function $\mathrm{MRL}(x)$ is in this case decreasing (starting from n/μ and approching $1/\mu$)

Hazard rate

- Consider a random interval whose length $X \sim \mathrm{Erl}(n,\mu)$.
  Assume that we know that the interval is longer than x.
  What is the probability that it will end in a short interval of length h after time x?

$$P\{X \le x+h \mid X > x\} = \frac{P\{x<X\le h\}}{P\{X>x\}} = \frac{f_X(x)h+o(h)}{1-F_X(x)}$$

- Thus, the hazard rate is

$$h(x) := \lim_{h\to 0} \frac{1}{h} P\{X \le x+h \mid X > x\} = \frac{f_X(x)}{1-F_X(x)} = \mu \cdot \frac{\frac{(\mu x)^{n-1}}{(n-1)!}}{\sum_{i=0}^{n-1} \frac{(\mu x)^i}{i!}}$$

- The hazard rate function $h(x)$ is in this case increasing (starting from 0 and approching μ)

Hyperexponential distribution

$$X \sim \mathrm{Hyp}(n, p_1, \mu_1, \ldots, p_n, \mu_n), \quad \mu_i > 0, \; p_i > 0, \; \sum_i p_i = 1$$

- parallel IID exponential phases; $X = I_1 X_1 + \ldots + I_n X_n$ where $X_i \sim \mathrm{Exp}(\mu_i)$ and $I_i \sim \mathrm{Bernoulli}(p_i)$ with $I_1 + \ldots + I_n = 1$
- n = total number of phases
- μ_i = intensity of phase i, p_i = probability of phase i

- Value space: $S_X = (0,\infty)$
- PDF and CDF:

$$f_X(x) = \sum_{i=1}^{n} p_i \mu_i e^{-\mu_i x}, \qquad x > 0$$

$$F_X(x) := P\{X \le x\} = \sum_{i=1}^{n} p_i (1 - e^{-\mu_i x})$$

Moments and the Laplace transform

$$X \sim \mathrm{Hyp}(n, p_1, \mu_1, \ldots, p_n, \mu_n), \quad \mu_i > 0, \; p_i > 0, \; \sum_i p_i = 1$$

- Mean value: $E[X] = E[I_1 X_1] + \ldots + E[I_n X_n] = p_1/\mu_1 + \ldots + p_n/\mu_n$
- 2nd moment: $E[X^2] = E[I_1 X_1{}^2] + \ldots + E[I_n X_n{}^2] = 2p_1/\mu_1{}^2 + \ldots + 2p_n/\mu_n{}^2$
- Variance: $D^2[X] = E[X^2] - E[X]^2 = \ldots$
- Standard deviation: $D[X] = \sqrt{D^2[X]} = \ldots$
- Coefficient of variation: $C[X] = D[X]/E[X] = \ldots \geq 1$
- Laplace transform : $E[e^{-sX}] = p_1(\mu_1/(\mu_1+s)) + \ldots + p_n(\mu_n/(\mu_n+s))$

Mean residual lifetime

- Consider a random interval whose length $X \sim \mathrm{Hyp}(n, p_1, \mu_1, \ldots, p_n, \mu_n)$.
  Assume that we know that the interval is longer than x.
  The mean residual lifetime is now

$$\mathrm{MRL}(x) := \frac{\int_x^{\infty} (1 - F_X(y))\, dy}{1 - F_X(x)} = \frac{\sum_{i=1}^{n} p_i \frac{1}{\mu_i} e^{-\mu_i x}}{\sum_{i=1}^{n} p_i e^{-\mu_i x}}$$

- The mean residual lifetime function $\mathrm{MRL}(x)$ is in this case increasing (starting from $p_1/\mu_1 + \ldots + p_n/\mu_n$ and approching $\max_i 1/\mu_i$)

Hazard rate

- Consider a random interval whose length $X \sim \mathrm{Hyp}(n, p_1, \mu_1, \ldots, p_n, \mu_n)$. Assume that we know that the interval is longer than x. The hazard rate is now

$$h(x) := \frac{f_X(x)}{1 - F_X(x)} = \frac{\sum_{i=1}^{n} p_i \mu_i e^{-\mu_i x}}{\sum_{i=1}^{n} p_i e^{-\mu_i x}}$$

- The hazard rate function $h(x)$ is in this case decreasing (starting from $p_1\mu_1 + \ldots + p_n\mu_n$ and approching $\min_i \mu_i$)

Pareto distribution

$$X \sim \text{Pareto}(b,\beta), \quad b>0,\ \beta>1$$

- – heavy tail distribution
- – b = location parameter
- – β = shape parameter

- Value space: $S_X = (0,\infty)$
- PDF and CDF:

$$f_X(x) = \beta b \left(\frac{1}{1+bx}\right)^{\beta+1}, \qquad x>0$$

$$F_X(x) := P\{X \le x\} = 1 - \left(\frac{1}{1+bx}\right)^{\beta}$$

Moments and the Laplace transform

$$X \sim \text{Pareto}(b, \beta), \quad b > 0, \beta > 1$$

- Mean value: $E[X] = \int_0^\infty \beta b x (1 + bx)^{-\beta-1}\, dx = 1/(b(\beta - 1))$ for $\beta > 1$
- Variance: $D^2[X] = \ldots$ for $\beta > 2$
- Standard deviation: $D[X] = \ldots$ for $\beta > 2$
- Coefficient of variation: $C[X] = C[X] = D[X]/E[X] = \ldots \geq 1$ for $\beta > 2$
- Laplace transform : $E[e^{-sX}] = \int_0^\infty \beta b\, e^{-sx} (1 + bx)^{-\beta-1}\, dx$ for $\beta > 1$

Mean residual lifetime

- Consider a random interval whose length $X \sim \text{Pareto}(b, \beta)$.
  Assume that we know that the interval is longer than x.
  The mean residual lifetime is now

$$\text{MRL}(x) := \frac{\int_x^\infty (1 - F_X(y))\, dy}{1 - F_X(x)} = \frac{1 + bx}{b(\beta - 1)}$$

- The mean residual lifetime function $\text{MRL}(x)$ is in this case linearly increasing (starting from $1/(b(\beta - 1))$ and approching ∞)

Hazard rate

- Consider a random interval whose length $X \sim \text{Pareto}(b, \beta)$.
  Assume that we know that the interval is longer than x.
  The hazard rate is now

$$h(x) := \frac{f_X(x)}{1 - F_X(x)} = \frac{b\beta}{1 + bx}$$

- The hazard rate function $h(x)$ is in this case decreasing (starting from $b\beta$ and approching 0)

Contents

- Basic concepts
- Discrete random variables
- Conditional expectation and variance
- Probability generating function (z-transform)
- Discrete distributions (count distributions)
- Continuous random variables
- Laplace transform
- Continuous distributions (time distributions)
- Other distributions and random variables

Uniform distribution

$$X \sim \mathrm{U}(a,b), \quad a < b$$

- continuous counterpart of “casting a die”

- Value space: $S_X = (a,b)$
- PDF:

$$f_X(x) = \frac{1}{b-a}, \quad x \in (a,b)$$

- CDF:

$$F_X(x) := P\{X \le x\} = \frac{x-a}{b-a}, \quad x \in (a,b)$$

- Mean value: $E[X] = \int_a^b x/(b-a)\, dx = (a+b)/2$
- Second moment: $E[X^2] = \int_a^b x^2/(b-a)\, dx = (a^2 + ab + b^2)/3$
- Variance: $D^2[X] = E[X^2] - E[X]^2 = (b-a)^2/12$

Standard normal (Gaussian) distribution

$$X \sim \mathrm{N}(0,1)$$

– limit of the "normalized" sum of IID r.v.s with mean 0 and variance 1

- Value space: $S_X = (-\infty,\infty)$
- PDF:

$$f_X(x) = \varphi(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

- CDF:

$$F_X(x) := P\{X \le x\} = \Phi(x) := \int_{-\infty}^{x} \varphi(y)\, dy$$

- Mean value: $E[X] = 0$
- Variance: $D^2[X] = 1$

Normal (Gaussian) distribution

$$X \sim \mathrm{N}(\mu, \sigma^2), \quad \mu \in \Re, \quad \sigma > 0$$

- if $(X - \mu)/\sigma \sim \mathrm{N}(0,1)$

- Value set: $S_X = (-\infty, \infty)$
- PDF:

$$f_X(x) = F_X'(x) = \tfrac{1}{\sigma}\varphi\left(\tfrac{x-\mu}{\sigma}\right)$$

- CDF:

$$F_X(x) := P\{X \le x\} = P\left\{\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right\} = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

- Mean value: $E[X] = \mu + \sigma E[(X - \mu)/\sigma] = \mu$
- Variance: $D^2[X] = \sigma^2 D^2[(X - \mu)/\sigma] = \sigma^2$

Properties

- (i) **Linear transformation**: Let $X \sim \mathrm{N}(\mu,\sigma^2)$ and $\alpha,\beta \in \Re$. Then

$$Y := \alpha X + \beta \sim \mathrm{N}(\alpha\mu + \beta, \alpha^2\sigma^2)$$

- (ii) **Sum**: Let $X_1 \sim \mathrm{N}(\mu_1,\sigma_1{}^2)$ and $X_2 \sim \mathrm{N}(\mu_2,\sigma_2{}^2)$ be **independent**. Then

$$X_1 + X_2 \sim \mathrm{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

- (iii) **Sample mean**: Let $X_i \sim \mathrm{N}(\mu,\sigma^2)$, $i = 1,\ldots,n$, be **IID**. Then

$$\overline{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i \sim \mathrm{N}(\mu, \tfrac{1}{n}\sigma^2)$$

Central limit theorem (CLT)

- Let $X_1,\ldots, X_n$ be **IID** with mean μ and variance σ^2 (and the third moment exists)
- **Central limit theorem**:

$$\frac{1}{\sigma/\sqrt{n}}(\bar{X}_n - \mu) \xrightarrow{\text{i.d.}} \mathrm{N}(0,1)$$

- It follows that for large values of n

$$\bar{X}_n \approx \mathrm{N}(\mu, \tfrac{1}{n}\sigma^2)$$

Other random variables

- In addition to discrete and continuous random variables, there are so called **mixed** random variables
  - containing some discrete as well as continuous portions
- **Example**:
  - The customer waiting time W in an M/M/1 queue has an **atom** at zero ($P\{W = 0\} = 1 - \rho > 0$) but otherwise the distribution is continuous