CALL BLOCKING IN ATM NETWORKS

- At the call level, from the point of view of resource allocation, even an variable bitrate (VBR) source is viewed as a constant bit source.
- Its rate = effective bandwidth.
- At the call level an ATM network can be analysed as a multi bitrate network (a mixture of different constant bitrate sources).

Our task is to find the (call level) blocking probability experienced by different connections.

- In the beginning we restrict ourselves to consider blocking in a single link.
- When a connection is set up, the amount of free capacity available on the link (in the case of a network, on all links along the route) must equal to or greater than the effective bandwidth of the connection,
  - if not, the call is rejected.
- Calculating blocking probabilities in a multi bitrate network (link) constitutes a multi dimensional generalization of the Erlang formula.

Suppose

- The capacity of the link is C
- The offered traffic consists of K different traffic classes $k = 1, \ldots, K$.
- The traffic of class k is characterized by
  - (effective) bandwidth b_k
  - arrival rate of calls λ_k (assumed to be Poissonian)
  - stopping intensity of calls μ_k (holding time is assumed to have exponential distribution)
  - traffic intensity $a_k = \lambda_k/\mu_k$

Denote the number of class-k calls in progress by N_k, $N_k = 0, 1, \ldots$.

- The state vector $\mathbf{N} = (N_1,\ N_2, \ldots, N_K)$ defines the set of calls in progress.
- A single state, i.e. point in the state space, is denoted by $\mathbf{n} = (n_1, \ldots, n_K)$.

State space

The possible states of a link with capacity C are constrained by the condition

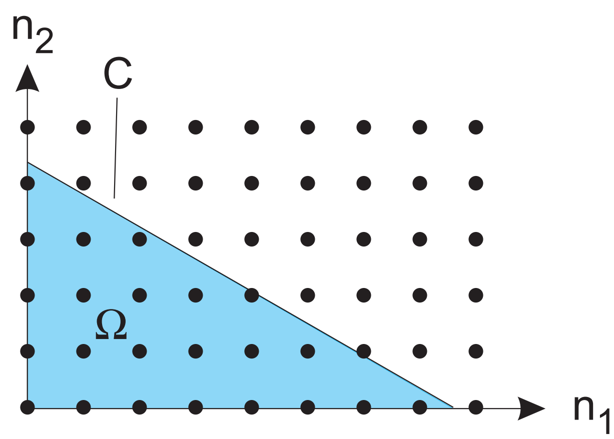
$$\sum_{k=1}^{K} n_k b_k \le C \qquad \text{or} \qquad \boxed{\mathbf{n} \cdot \mathbf{b} \le C}$$

where we have denoted

$$\begin{cases} \mathbf{b} = (b_1, \ldots, b_K) \\ \mathbf{n} \cdot \mathbf{b} = \sum_{k=1}^{K} n_k b_k \end{cases}$$

The state space of the link Ω consists of the allowed states

$$\Omega = \{\mathbf{n} : \mathbf{n} \cdot \mathbf{b} \le C\}$$



Blocking states

The set of blocking states $\mathcal{B}_k$ of traffic class k consists of those allowed states where no more class-k calls can be admitted without violating the capacity constraint

$$\boxed{\mathcal{B}_k = \{\mathbf{n} : C - b_k < \mathbf{n}\cdot\mathbf{b} \le C\}}$$

The greater b_k, the larger is the set of blocking states.

Denote by $\mathbf{e}_k$ the vector with all components equal to zero, except for component k which is one. By means of this, the set of blocking states can be written as

$$\boxed{\mathcal{B}_k = \{\mathbf{n} : \mathbf{n} \in \Omega,\ \mathbf{n} + \mathbf{e}_k \notin \Omega\}}$$

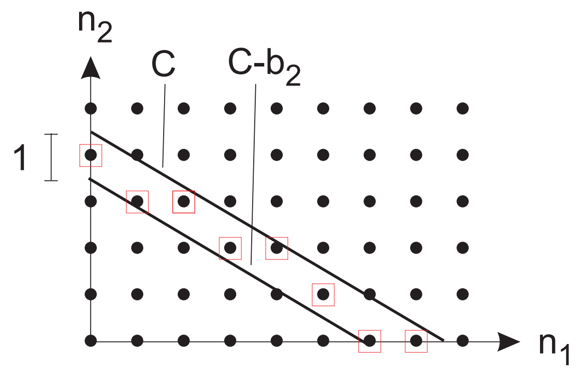
A new class-k call would take the system outside the allowed region.

Blocking states (continued)

The figures shows the blocking states of classes 1 and 2.

The set of blocking states $\mathcal{B}_k$ can be characterized as:

- "Last states" at the end of columns is the k-direction
- States between the hyperplanes corresponding to the capacity occupancies C and $C - b_k$
  - the latter is obtained by translating the C-hyperplane by one step in the k-direction towards the origin.



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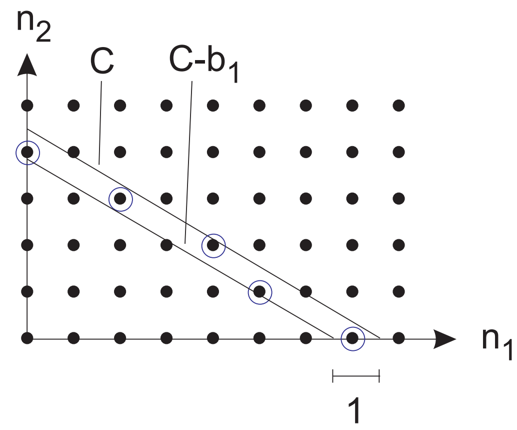

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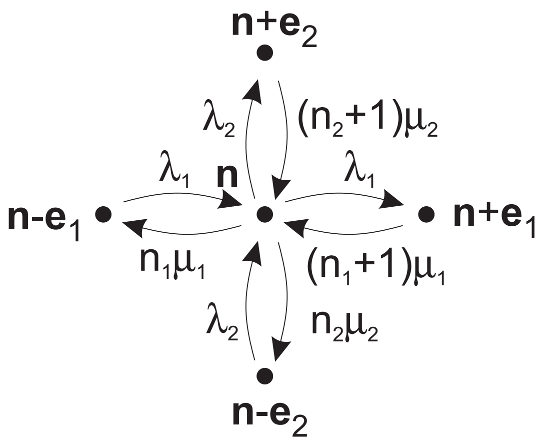
State probabilities

Under the assumptions made the system constitutes a Markov process

- multi dimensional birth-death process
- the state transition from an to a typical state within the allowed region are shown in the figure



The balance equations have a product form solution for $\pi(\mathbf{n}) = \mathrm{P}\{\mathbf{N} = \mathbf{n}\}$,

$$\boxed{\pi(\mathbf{n}) = G(\Omega)^{-1} \cdot \frac{a_1^{n_1}}{n_1!} \cdots \frac{a_K^{n_K}}{n_K!} = G(\Omega)^{-1} \cdot \prod_{k=1}^{K} \frac{a_k^{n_k}}{n_k!}}$$

where $G(\Omega)$ is the normalization constant

$$G(\Omega) = \sum_{\mathbf{n}\in\Omega} \frac{a_1^{n_1}}{n_1!} \cdots \frac{a_K^{n_K}}{n_K!}$$

In the sequel, we denote by $G(\mathcal{S})$ the sum of the unnormalized state probabilities (so called state sum) over any given set of states $\mathcal{S}$.

The product form solution applies, since it satisfies even the detailed balance condition, i.e. the probability flows between any two neighbouring states $\mathbf{n} - \mathbf{e}_k$ and $\mathbf{n}$ are in balance

$$\lambda_k \pi(\mathbf{n} - \mathbf{e}_k) = n_k \mu_k \pi(\mathbf{n}) \qquad \text{since} \qquad \pi(\mathbf{n}) = \frac{a_k}{n_k} \pi(\mathbf{n} - \mathbf{e}_k)$$

Blocking probabilities

The blocking probability B_k of class-k calls is the probability that the system is the set of blocking states of that class, i.e. the probability of the set $\mathcal{B}_k$.

$$\boxed{B_k = \sum_{\mathbf{n}\in\mathcal{B}_k} \pi(\mathbf{n}) = \frac{G(\mathcal{B}_k)}{G(\Omega)}}$$

Reciprocity of blocking probabilities

By derivating the expression for $G(\Omega)$ we find

$$\frac{\partial}{\partial a_k}G(\Omega) = \frac{\partial}{\partial a_k}\sum_{\mathbf{n}\in\Omega}\frac{a_1^{n_1}}{n_1!}\cdots\frac{a_K^{n_K}}{n_K!} = \sum_{\mathbf{n}\in\Omega}\frac{a_1^{n_1}}{n_1!}\cdots\frac{a_k^{n_k-1}}{(n_k-1)!}\cdots\frac{a_K^{n_K}}{n_K!} = \sum_{\mathbf{n}\in\Omega-\mathcal{B}_k}\frac{a_1^{n_1}}{n_1!}\cdots\frac{a_K^{n_K}}{n_K!} = G(\Omega)-G(\mathcal{B}_k)$$

By calculating $G(\mathcal{B}_k)$ from this and inserting into the expression of B_k we get

$$B_k = 1 - \frac{1}{G(\Omega)}\frac{\partial}{\partial a_k}G(\Omega) = 1 - \frac{\partial}{\partial a_k}\log G(\Omega)$$

By partial derivation it follows

$$\frac{\partial B_j}{\partial a_i} = -\frac{\partial}{\partial a_i}\frac{\partial}{\partial a_j}\log G(\Omega) = -\frac{\partial}{\partial a_j}\frac{\partial}{\partial a_i}\log G(\Omega) = \frac{\partial B_i}{\partial a_j} \qquad \boxed{\frac{\partial B_j}{\partial a_i} = \frac{\partial B_i}{\partial a_j}}$$

Truncation of the state space of a time reversible system

Let Ω be the state space of a time reversible system. Time reversibility is equivalent to saying that the detailed balance is valid

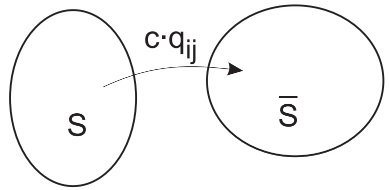
$$\pi_i q_{i,j} = \pi_j q_{j,i}, \qquad \forall i, j \in \Omega$$

Consider a modified system, where the state space is divided into two parts

$$\Omega = S + \overline{S}$$

and the transition rates are otherwise the same as before but the rate of a transition $i \to j$ is multiplied by c when $i \in S$ and $j \in \overline{S}$:

$$\begin{cases} q'_{i,j} = c\, q_{i,j}, & i \in S,\ j \in \overline{S} \\ q'_{i,j} = q_{i,j}, & \text{otherwise} \end{cases}$$



Proposition: The equilibrium probabilities of the modified system are

$$\pi'_i = \begin{cases} \dfrac{1}{G}\,\pi_i, & \text{when } i \in S \\ \dfrac{1}{G}\,c\,\pi_i, & \text{when } i \in \overline{S} \end{cases}$$

where G is the normalization constant
$G = \sum_{i \in S} \pi_i + \sum_{i \in \overline{S}} \pi_i$

Truncation of the state space (continued)

Proof: The state probabilities π'_i given in the proposition satisfy the detailed balance conditions of the modified system

$$\pi'_i q'_{i,j} = \pi'_j q'_{j,i}, \qquad \forall i, j \in \Omega$$

Since detailed balance implies (is a sufficient condition to) global balance, the π'_i satisfy the (global) balance conditions of the modified system, and thus they provide the unique solution for the equilibrium probabilities.

Corollary 1. By setting $c = 0$ (truncation of the state space), we see that the state probabilities of the truncated system (state space S) are up to a normalization constant the same as in the original system

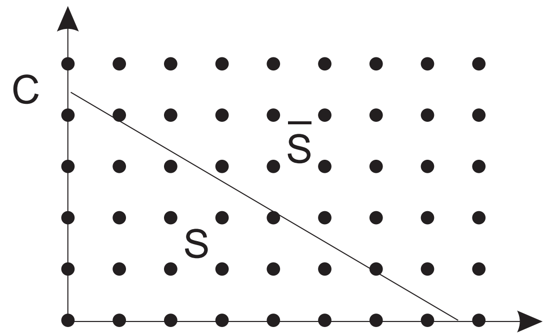
$$\pi'_i = \frac{1}{G}\pi_i, \qquad G = \sum_{i \in S} \pi_i$$

Corollary 2. The following state distributions hold:

- Erlang's loss system: truncated Poisson distribution
- Engset's system: truncated binomial distribution
- $M/M/1/m$ queue: truncated geometric distribution

Truncation of the state space (continued)

Corollary 3. In a multi bitrate system, where the capacity is shared by several traffic streams, the state probabilities are of the product form (product of Poisson distributions).



Product form solution is trivially true for an infinite capacity system, where the traffic streams are independent. By the truncation principle, the same distribution applies in the truncated state space.

Corollary 4. If m $M/M/1$ queues share a common waiting room (in total, there are K system places, including the common waiting room and the servers of the queues), then the joint probability distribution is

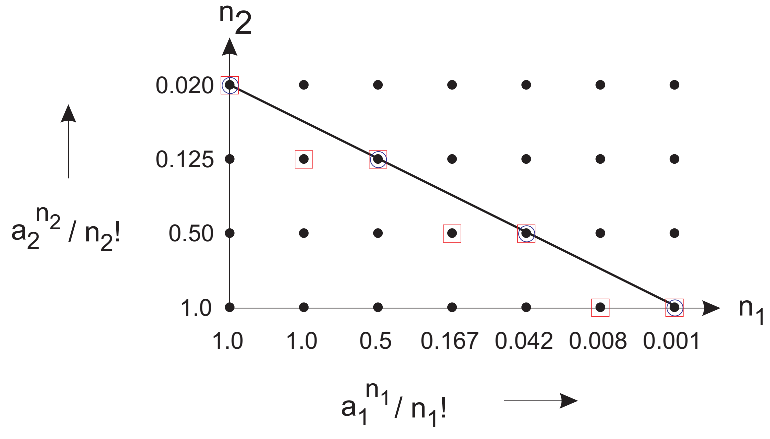
$$\mathrm{P}\{N_1 = n_1, \ldots, N_m = n_m\} = \frac{1}{G}\,\rho_1^{n_1} \cdots \rho_m^{n_m}, \qquad \text{when } n_1 + \cdots + n_m \leq K$$

where ρ_i is the load of queue i. The result follows as in the case of an infinite waiting room, the queues are independent ant the joint distribution is trivially the product of factors $(1-\rho_i)\rho_i^{n_i}$.

Example of calculating the blocking probabilities

Suppose a link with capacity C is used by connections belonging to two different classes. The parameters are:

$$C = 6 \text{ Mbps} \qquad \begin{cases} a_1 = 1 \\ a_2 = 0.5 \end{cases} \qquad \begin{cases} b_1 = 1 \text{ Mbps} \\ b_2 = 2 \text{ Mbps} \end{cases}$$



The state space, the allowed states and the blocking states of the traffic classes are shown in the figure.

The marginal distributions are $a_k^{n_k}/n_k!$, $k = 1, 2$. The constant factor $\exp(-a_k)$ can be omitted because after the truncation we anyway have to normalize the result.

The unnormalized probability of a single state is the product of the marginal values.

The normalization factor $G(\Omega)$ is the sum of the unnormalized probabilities, $G(\Omega) = 4.41$.

The sums of the unnormalized probabilities over the sets of blocking states are

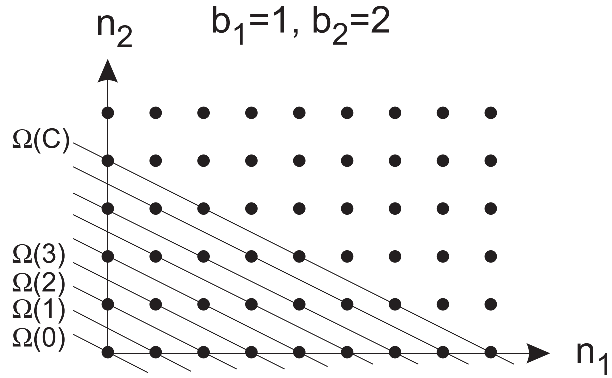
$$G(\mathcal{B}_1) = 0.11, \qquad G(\mathcal{B}_2) = 0.32$$

from which follow the blocking probabilities

$$B_1 = \frac{G(\mathcal{B}_1)}{G(\Omega)} = 2.4\% \qquad B_2 = \frac{G(\mathcal{B}_2)}{G(\Omega)} = 7.3\%$$

The Kaufman-Roberts recursion

In the previous example was based on "picking by hand" each allowed state or blocking state and summing the probabilities. There are more powerful methods to do the calculation. The Kaufman-Roberts recursion provides one such method.



In this method we consider partial state sums over certain subsets of the state space. In particular we define the set $\Omega(c)$ to consist precisely of those states where the occupied capacity equals c,

$$\mathbf{n} \in \Omega(c) \Leftrightarrow \mathbf{n} \cdot \mathbf{b} = c$$

These define a partition of the state space, $\Omega = \bigcup_{c=0}^{C} \Omega(c)$.

The sum of the unnormalized probabilities over the set $\Omega(c)$ is $G(\Omega(c))$. For brevity, we denote it by $G(c)$.

$$G(c) = \sum_{\mathbf{n}\cdot\mathbf{b}=c} \frac{a_1^{n_1}}{n_1!} \cdots \frac{a_K^{n_K}}{n_K!}$$

The following recursion is valid for $G(c)$ (the proof is left as an exercise)

$$\boxed{c\,G(c) = \sum_{k=1}^{K} a_k b_k G(c - b_k)}$$

Initialization: $G(0) = 1$, $G(c) = 0$, when $c < 0$.

The Kaufman-Roberts recursion (continued)

Since the $\Omega(c)$ constitute a partition of Ω, the normalization constant is

$$G = G(\Omega) = \sum_{c=0}^{C} G(c)$$

Thus the distribution of the capacity occupancy is

$$\mathrm{P}\{\mathbf{N} \in \Omega(c)\} = \mathrm{P}\{\mathbf{N} \cdot \mathbf{b} = c\} = \frac{G(c)}{G} \ ,$$

Clearly, blocking states for class-k calls are the states, where the free capacity is less than b_k, that is where the capacity occupancy is within the range $(C - b_k + 1) \ldots, C$.

$$\mathcal{B}_k = \bigcup_{c=C-b_k+1}^{C} \Omega(c)$$

and correspondingly the blocking probability is

$$\boxed{B_k = \frac{G(\mathcal{B}_k)}{G(\Omega)} = \frac{\sum\limits_{c=C-b_k+1}^{C} G(c)}{\sum\limits_{c=0}^{C} G(c)}}$$

The Kaufman-Roberts recursion (continued)

Example. We apply the KR-recursion to the problem of the previous example.
We use 1 Mbps as the unit of capacity.

$$C = 6 \qquad \begin{cases} a_1 = 1 \\ a_2 = 0.5 \end{cases} \qquad \begin{cases} b_1 = 1 \\ b_2 = 2 \end{cases}$$

Recursion and its initialization

$$G(c) = \frac{1}{c}(G(c-1) + G(c-2)) \qquad \begin{cases} G(0) = 1 \\ G(c) = 0, \quad \text{when } c < 0 \end{cases}$$

$$\begin{aligned} G(1) &= \tfrac{1}{1}(G(0) + G(-1)) = 1 & G(4) &= \tfrac{1}{4}(G(3) + G(2)) = \tfrac{5}{12} \\ G(2) &= \tfrac{1}{2}(G(1) + G(0)) = 1 & G(5) &= \tfrac{1}{5}(G(4) + G(3)) = \tfrac{13}{60} \\ G(3) &= \tfrac{1}{3}(G(2) + G(1)) = \tfrac{2}{3} & G(6) &= \tfrac{1}{6}(G(5) + G(4)) = \tfrac{19}{180} \end{aligned}$$

$$B_1 = \frac{G(6)}{G(0) + \cdots + G(6)} = \frac{19}{793} \approx 2.4\%, \qquad B_2 = \frac{G(5) + G(6)}{G(0) + \cdots + G(6)} = \frac{58}{793} \approx 7.3\%$$

Convolution method

Instead of the Kaufman-Roberts recursion the capacity occupancy distribution can also be computed by convolutions.

To this end, calculate first the capacity occupancy distribution for individual streams.

- The probabilities need not be normalized, since at the end we anyway have to normalize the result.
- The distribution of the number of connections of type k, n_k, is (unnormalized) $a_k^{n_k}/n_k!$
- Correspondingly, $a_k^{n_k}/n_k!$ is the (unnormalized) probability that the capacity occupancy of class-k calls is $n_k b_k$.
- The probability is non-zero only for such occupancies, which are integer multiples of b_k.

For the previous example, the unnormalized probabilities of the capacity occupancy of traffic classes 1 and 2, up to occupancy value $c = 6$, are

$$\begin{cases} p_1 = (1.0 \;\; 1.0 \;\; 0.5 \;\; 0.167 \;\; 0.042 \;\; 0.008 \;\; 0.001) \\ p_2 = (1.0 \;\; 0 \;\; 0.5 \;\; 0 \;\; 0.125 \;\; 0 \;\; 0.020) \end{cases} \qquad \text{notice the zeros in between}$$

Convolution method (continued)

The total occupancy of the link is the sum of the occupancies of individual streams.

- In the case of infinite capacity, the streams are independent.
- The distribution of the sum of independent random variables is obtained by convolution
- The finiteness of the capacity C only implies that the probabilities will be zero for occupancies $> C$ and that the probabilities have to be renormalized.

Denote $G = (G(0)\ G(1)\ \cdots\ G(C))$. Then, G will be obtained by convolution,

$$G = p_1 \otimes p_2 \qquad \Leftrightarrow \qquad G(c) = \sum_{i+j=c} p_1(i)p_2(j) = \sum_{i=0}^{c} p_1(i)p_2(c-i)$$

In the case of several traffic classes, one adds (convolutes) one class at a time:

$$G_2 = p_2 \otimes p_1,\ G_3 = p_3 \otimes G_2,\ \cdots\ ,\ G_{K-1} = p_{K-1} \otimes G_{K-2},\ G = G_K = p_K \otimes G_{K-1}$$

As before, at the end the result has to be normalized

$$\mathrm{P}\{\mathbf{N} \cdot \mathbf{b} = c\} = \frac{G(c)}{\sum_{i=0}^{C} G(i)}, \quad c = 0, \ldots, C$$

Convolution method (continued)

Convolutions can also be calculated by generating functions

$$\begin{cases} p_k(n_k) = \dfrac{a_k^{n_k}}{n_k!} \\ \\ P_k(z) = \sum\limits_{n_k=0}^{\infty} p_k(n_k)\, z^{n_k b_k} = e^{a_k z^{b_k}} \end{cases}$$

(generating function for the capacity occupancy distribution;
the occupancy in the state n_k is $n_k b_k$)

The generating function for the total occupancy is

$$\sum_{c=0}^{\infty} G(c) z^c = \prod_{k=1}^{K} P_k(z) = \exp\left(\sum_k a_k z^{b_k}\right)$$

The (unnormalized) probabilities of individual occupancy states, $G(c)$, can be identified from the coefficients of different powers of z.

In practice, one takes from each $P_k(z)$ as many terms as needed from the beginning (up to power C) and multiplies these polynomials. The coefficients of different powers of z in the resulting product polynomial give the unnormalized probabilities.

Multiplication of polynomials $\equiv$ convolution:

$$\sum_i p_1(i) z^i \sum_j p_2(j) z^j = \sum_k \left(\sum_{i=0}^{k} p_1(i) p_2(k-i) \right) z^k$$

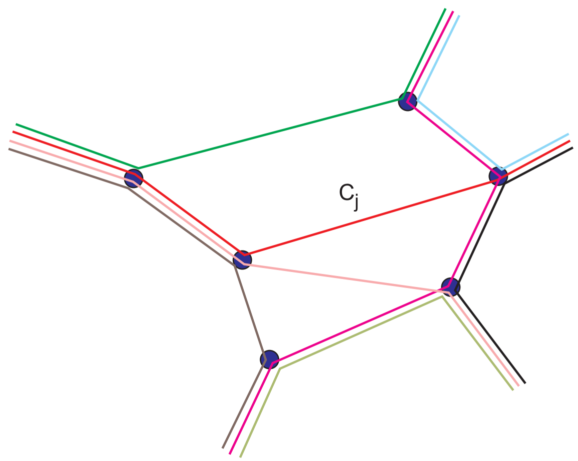
End-to-end blocking probability in a multi bitrate network

Above we considered blocking in a single link with a given capacity C. Now we consider the calculation of end-to-end blocking probability in a network, consisting of several links $j = 1, \ldots, J$, with capacities C_j.

As we shall see, conceptually the case of a network implies only a relatively small generalization to the single link case.

- Another matter is that, in practice, the exact calculation of the end-to-end blockings for a real network may require a big (sometimes prohibitively large) effort.
- Earlier we presented an approximate method (reduced load approximation) for calculating the end-to-end blocking probabilities in a single traffic class (telephone) network. This approximate method can be extended to the case of multi bitrate traffic as well, though we do not present the extension here.
- The exact evaluation of blocking probabilities in a network with a single traffic class is a special case of the multi bitrate system to be considered in the following.

End-to-end blocking probability in a multi bitrate network (continued)



- Links $j = 1, \ldots, J$, capacities C_j
- Traffic classes $k = 1, \ldots, K$
- A traffic class is defined by the route of a connection and its required capacity.
- A class-k call requires $b_{j,k}$ capacity units (trunks) on link j; the effective capacity depends on the capacity of the link and thus may vary from one link to another.
- Offered traffic intensity of class-k calls is a_k.

The system is again described by the numbers of calls in progress in different traffic classes. A point in the state space is the vector

$$\mathbf{n} = (n_1, \ldots, n_K)$$

Define the capacity vector on link j to have components which tell the capacities of calls in different traffic classes required on that link ($= 0$, if the connection does not use the link)

$$\mathbf{b}_j = (b_{j,1}, \ldots, b_{j,K})$$

In the state $\mathbf{n}$, $\mathbf{n} \cdot \mathbf{b}_j$ capacity units are occupied on link j.

End-to-end blocking probability in a multi bitrate network (continued)

If the link capacities were infinite, all traffic classes would be independent and the number of calls in each class would obey a Poisson distribution

$$p_k(n_k) = \mathrm{P}\{N_k = n_k\} = \frac{a_k^{n_k}}{n_k!}\, e^{-a_k}$$

The exponential factor (constant) is not important and could be omitted; normalization will anyway be done at the end.

Correspondingly, the distribution of all calls in the network would be

$$p(\mathbf{n}) = \mathrm{P}\{N_1 = n_1, \ldots, N_1 = n_1\} = \prod_{k=1}^{K} p_k(n_k)$$

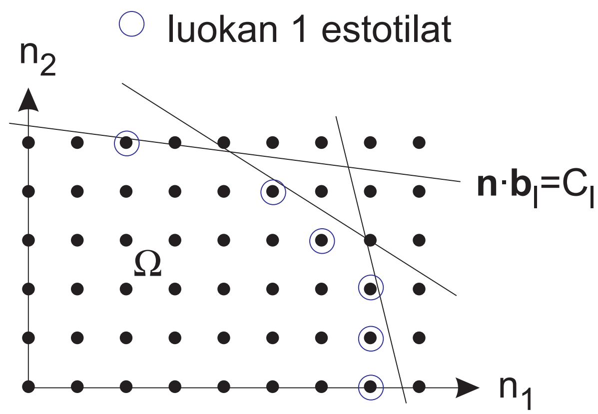
It is easy to see that this distribution satisfies the detailed balance condition.

- Thus, even when the links have finite capacities and the state space is truncated accordingly, the distribution remains the same, up to the normalization, in the truncated the state space.

The finite link capacities constrain the state space Ω:

$$\mathbf{n} \cdot \mathbf{b}_j \leq C_j, \forall j$$

The difference in comparison with the single link case is that now we have several constraints. Each connection consumes several resources of the network.



End-to-end blocking probability in a multi bitrate network (continued)

In the region of allowed states Ω the state probabilities are still of the product form,

$$\pi(\mathbf{n}) = \begin{cases} \frac{1}{G} p(\mathbf{n}), & \mathbf{n} \in \Omega \\ 0, & \mathbf{n} \notin \Omega \end{cases} \qquad \text{where} \quad G = G(\Omega) = \sum_{\mathbf{n}\in\Omega} p(\mathbf{n})$$

The set of blocking states for traffic stream k is

$$\mathcal{B}_k = \{\mathbf{n} : \mathbf{n} \in \Omega,\ \mathbf{n} + \mathbf{e}_k \notin \Omega\}$$

(the blocking states are the last states within Ω when moving in the k-direction)

and the blocking probability of class-k calls is

$$B_k = \sum_{\mathbf{n}\in\mathcal{B}_k} \pi(\mathbf{n}) = \frac{\sum_{\mathbf{n}\in\mathcal{B}_k} p(\mathbf{n})}{\sum_{\mathbf{n}\in\Omega} p(\mathbf{n})} = \frac{G(\mathcal{B}_k)}{G(\Omega)}$$

- Formally, the solution is precisely the same as in the case of a single link.
- Due to several constraints, however, the sets Ω and $\mathcal{B}_k$ are complex and it is not easy to evaluate the state sums.
- The size of the state space for any realistic system is astronomical, and calculating the sums state by state is not feasible.

End-to-end blocking probability in a multi bitrate network (continued)

It is, however, useful to note that in order to calculate the blocking probability it is sufficient to be able to calculate state sums of type $G(\Omega)$, where Ω is a state space with linear constraints.

Let us define the set of non-blocking states $\mathcal{S}_k$ for traffic class k

$$\mathcal{S}_k = \{\mathbf{n} : \mathbf{n} + \mathbf{e}_k \in \Omega\} = \{\mathbf{n} : \mathbf{n} \cdot \mathbf{b}_j \leq C_j - b_{j,k} \ \forall j\}$$

states $\mathbf{n}$ where a new class-k call can be admitted

This is the complement of the set $\mathcal{B}_k$ with respect to Ω: $\Omega = \mathcal{B}_k + \mathcal{S}_k$. Correspondingly, we have

$$G(\Omega) = G(\mathcal{B}_k) + G(\mathcal{S}_k) \quad \Rightarrow \quad B_k = \frac{G(\mathcal{B}_k)}{G(\Omega)} = 1 - \frac{G(\mathcal{S}_k)}{G(\Omega)}$$

The definition of the set S_k is by form similar as that of the set Ω, with the difference that the capacity of each link j has been reduced by the amount $b_{j,k}$.

Define the vectors $\mathbf{C} = (C_1, \ldots, C_J)$ ja $\mathbf{b}^{(k)} = (b_{1,k}, \ldots, b_{J,k})$.

Vector $\mathbf{C}$ defines the set Ω: $\Omega = \Omega(\mathbf{C})$. With this notation we have $\mathcal{S}_k = \Omega(\mathbf{C} - \mathbf{b}^{(k)})$. Using another shorthand notation, $G(\Omega(\mathbf{C})) = G(\mathbf{C})$, we finally obtain

$$\boxed{B_k = 1 - \frac{G(\mathbf{C} - \mathbf{b}^{(k)})}{G(\mathbf{C})}}$$

Calculating blocking probabilities in a hierarchical access network

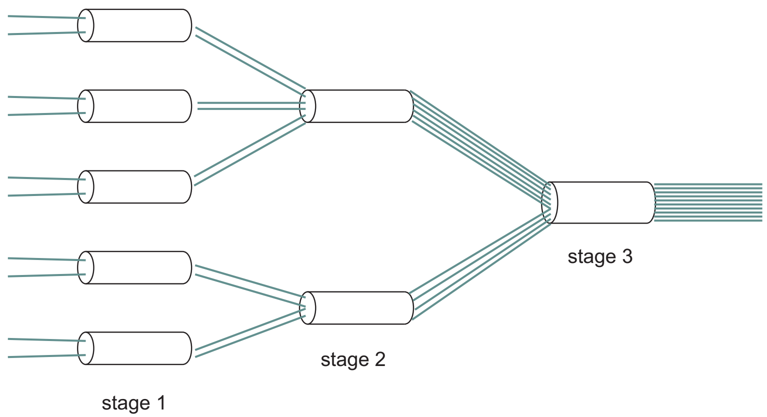
Evaluating the exact formula for the end-to-end blocking, in general, is not feasible due to the huge size of the state space.

No general method is either available for speeding up the computation so that the exact value could always be computed

- Approximate methods, however, exist. In addition to the reduced load approximation, Monte Carlo simulation provides a tool for estimating the sums; then the results contain a statistical error.

In some special cases, it is possible to calculate the sums exactly with a reasonable effort.

One such case is, when the topology of the network is a tree, as in an access network depicted in the figure.



Blocking probabilities in a hierarchical access network (continued)

The idea is to proceed in stages:

- First one calculates the distribution of the capacity occupancy in the links of stage 1, assuming the capacities of all the links of the upper stages to be infinite.
- Then the links of stage 1 are independent. The capacity occupancy distribution of each link can be calculated as shown before by convolving the distributions of different classes offered to the link.
  - Assuming, for a while, also the capacities of the stage 1 links infinite, the capacity occupancy distribution is the convolution of the distributions of individual classes.
  - The joint process satisfies the detailed balance, whence the finite capacity only implies truncation of the distribution obtained from the convolution (the normalization of the result is deferred to a later phase). Also the truncated joint process satisfies the detailed balance condition.
- Consider now one of the links of stage 2. As the stage 1 links feeding this link are independent, the offered capacity occupancy distribution is obtained by convolving the truncated distributions of the feeding links. The joint process satisfies again the detailed balance, whence the finite capacity of the stage 2 link is taken into account by truncating the distribution resulting from the convolutions (the capacities of all links of the higher stages are still assumed infinite).

Blocking probabilities in a hierarchical access network (continued)

- In the same way all links of stage 2 are handled by calculating the capacity occupancy distributions truncated according to the capacity constraints.
- Then we proceed to the next stage and repeat the procedure.

In handling any given link, two operations are involved:

1. The truncated occupancy distributions of the links of the previous stage are convolved.
2. The resulting distribution is truncated according to the capacity of the link.

This convolution-truncation-recursion is carried through the whole network, proceeding form the lower stages towards the highest stage. Finally, we end up with the highest link, where the procedure gives the truncated "capacity occupancy distribution", i.e. the values $g(c) = \mathrm{P}\{\text{cap. occ. of the link} = c\}$, $c = 0, \ldots, C$, where C is the capacity of the considered link.

If $g(c)$ were the real distribution, then the sum would be $\Sigma_{c=0}^{C} g(c) = 1$. However, we have not normalized the distribution at any stage, an note that, in fact, we have obtained the sum of the state probabilities of the original infinite capacity system over all the states that belong to the allowed state space. The truncations made at each stage have pruned out all the states that are outside the allowed region.

Calculating blocking probabilities in a hierarchical access network

Thus, we deduce that the following holds

$$\boxed{G(\Omega) = G(\mathbf{C}) = \sum_{c=0}^{C} g(c)}$$

$G(\mathbf{C})$ is the shorthand notation for $G(\Omega(\mathbf{C}))$, where we have emphasized the dependence of the state space on the capacities of different links of the network (i.e. on the vector $\mathbf{C}$).

In the case of the access network we know how to calculate state sums of the type $G(\mathbf{C})$. As said before that is all that is needed for the calculation of the end-to-end blocking probability of a given traffic class,

$$\boxed{B_k = 1 - \frac{G(\mathbf{C} - \mathbf{b}^{(k)})}{G(\mathbf{C})}}$$

The state sum is calculated twice: first using the original full link capacities, yielding $G(\mathbf{C})$, and then for a network, where the capacity of each link j along the route of the considered traffic class k are reduced by the amount $b_{j,k}$, yielding $G(\mathbf{C} - \mathbf{b}^{(k)})$.

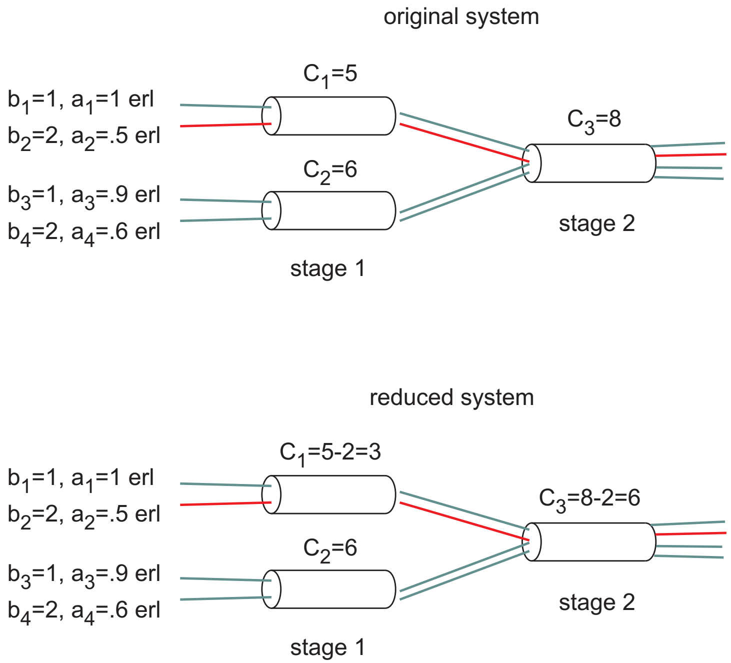
Blocking probabilities in an access network: example

Consider a two stage access network shown in the figure.

Both first stage links (links 1 and 2) are offered two traffic classes, bandwidths $b_1 = b_3 = 1$ and $b_2 = b_4 = 2$ with traffic intensities (erl) $a_1 = 1$, $a_2 = 0.5$, $a_3 = 0.9$ and $a_4 = 0.6$.

The task is to calculate the blocking probability of calls in class 2 offered to link 1.

Thus we have to calculate the state sum both for the original system and a system, where the link capacities along the route have been reduced by 2 (lower figure).



The calculations can be done expediently by means of the generating functions.

Denote by T_n a truncation operator, that takes from a polynomial (or power series) of z the terms up to the term z^n. Example

$$T_3(1 + z + 0.3z^2 + 0.1z^3 + 0.05z^4 + 0.01z^5) = 1 + z + 0.3z^2 + 0.1z^3$$

Example (continued)

Original system

Truncated generating functions of the (unnormalized) occupancy distributions arising from the traffic streams offered to link 1

$$\begin{cases} g_{1,1}(z) = T_5(\sum_{n=0}^{\infty} \frac{(1.0z)^n}{n!}) = 1 + z + 0.5z^2 + 0.1667z^3 + 0.0417z^4 + 0.0083z^5 \\ g_{1,2}(z) = T_5(\sum_{n=0}^{\infty} \frac{(0.5z^2)^n}{n!}) = 1 + 0.5z^2 + 0.125z^4 \end{cases}$$

The generating function of the total (unnormalized) occupancy distribution of link 1

$$g_1(z) = T_5(g_{1,1}(z)g_{1,2}(z)) = 1 + z + z^2 + 0.6667z^3 + 0.4167z^4 + 0.2167z^5$$

Similarly for link 2

$$\begin{cases} g_{2,1}(z) = T_6(\sum_{n=0}^{\infty} \frac{(0.9z)^n}{n!}) = 1 + 0.9z + 0.4050z^2 + 0.1215z^3 + 0.0273z^4 + 0.0049z^5 + 0.0007z^6 \\ g_{2,2}(z) = T_6(\sum_{n=0}^{\infty} \frac{(0.6z^2)^n}{n!}) = 1 + 0.6z^2 + 0.18z^4 + 0.036z^6 \end{cases}$$

$$g_2(z) = T_6(g_{2,1}(z)g_{2,2}(z)) = 1 + 0.9z + 1.0050z^2 + 0.6615z^3 + 0.4503z^4 + 0.2398z^5 + 0.1260z^6$$

The generating function of the total (unnormalized) occupancy distribution on link 3

$$g_3(z) = T_8(g_1(z)g_2(z)) = 1 + 1.9z + 2.905z^2 + 3.2332z^3 + 3.1335z^4 + 2.6133z^5 + 1.8709z^6 + 1.1595z^7 + 0.6169z^8$$

Example (continued)

Reduced system (denote the corresponding generating functions by h)

The capacity of link 1 in the reduced system is 3.

$$\begin{cases} h_{1,1}(z) = T_3(\sum_{n=0}^{\infty} \frac{(1.0z)^n}{n!}) = 1 + z + 0.5z^2 + 0.1667z^3 \\ h_{1,2}(z) = T_3(\sum_{n=0}^{\infty} \frac{(0.5z^2)^n}{n!}) = 1 + 0.5z^2 \end{cases}$$

$$h_1(z) = T_3(h_{1,1}(z)h_{1,2}(z)) = 1 + z + z^2 + 0.6667z^3$$

Link 2 in the reduced system is the same as in the original system:

$$h_2(z) = g_2(z) = 1 + 0.9z + 1.0050z^2 + 0.6615z^3 + 0.4503z^4 + 0.2398z^5 + 0.1260z^6$$

The capacity of link 3 in the reduced system is 6.

$$h_3(z) = T_6(h_1(z)h_2(z)) = 1 + 1.9z + 2.905z^2 + 3.2332z^3 + 2.7168z^4 + 2.0217z^5 + 1.2572z^6$$

Blocking probability (traffic class $k = 2$: calls of bandwidth 2 offered to link 1)

$$B_2 = 1 - \frac{G(\mathbf{C} - \mathbf{b}^{(2)})}{G(\mathbf{C})} = 1 - \frac{h_3(1)}{g_3(1)} = \underline{0.184}$$