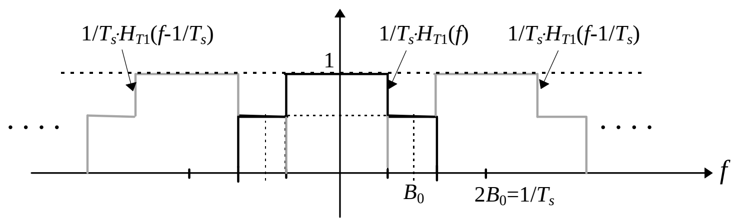
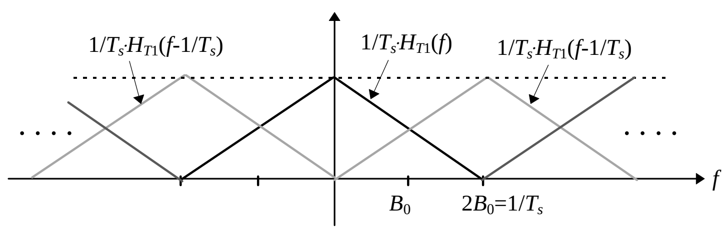
S-38.211 Signal Processing in Communications	Stefan Werner
Solutions to Exercise 1, October 8, 1997	phone: 451 2437 room: SG224

Solution E 1-1

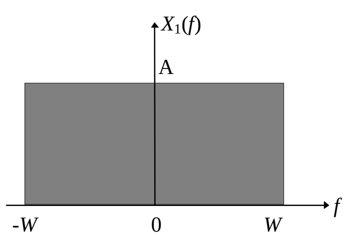
a) According to the Nyquist criterion the following must hold: $\frac{1}{T_s}\sum_{m=-\infty}^{\infty} H(f - m/T_s) = 1$

Let us do this graphically! Take the original spectrum and scale it with $1/T_s$.





b) When solving for the time-domain waveform it is nice to have a good transformation table that covers most of the Fourier transforms. However, if you are not so familiar with Fourier transforms, here are some different solutions. Let us first study the rectangular spectrum given by the figure below.



Its time-domain waveform is given by

$$x_1(t) = \int_{-\infty}^{\infty} X_1(f) e^{j2\pi ft} df = \int_{-W}^{W} A e^{j2\pi ft} df = A \frac{e^{j2\pi ft}}{2j\pi t}\bigg|_{f=-W}^{W}$$

$$= A\frac{e^{j2\pi Wt} - e^{-j2\pi Wt}}{2j\pi t} = A\frac{\sin 2\pi Wt}{\pi t} = 2AW\mathrm{sinc}(2Wt)$$

We can now easily calculate the pulse waveform $h_{T1}(t)$ by first recognizing that the spectrum is a sum of two rectangular spectra and use the result above. We get

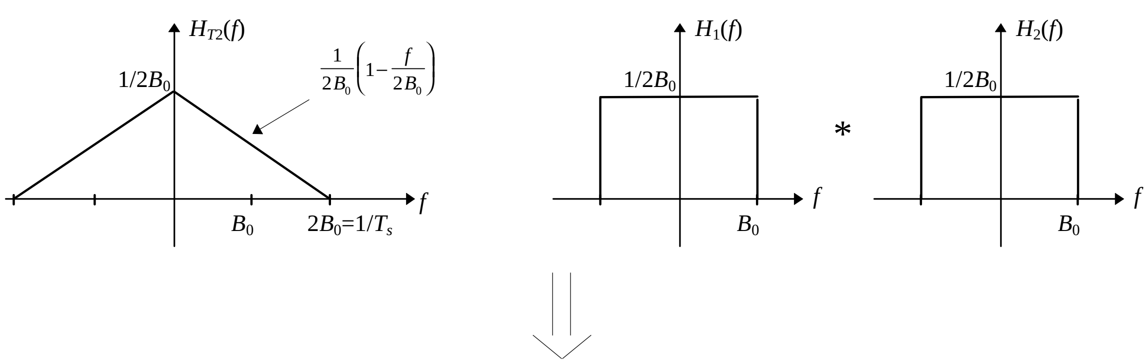
$$h_{T1}(t) = \int_{-\infty}^{\infty} H_{T1}(f) e^{j2\pi ft} df = \int_{-2B_0/3}^{2B_0/3} \frac{1}{4B_0} e^{j2\pi ft} df + \int_{-4B_0/3}^{4B_0/3} \frac{1}{4B_0} e^{j2\pi ft} df$$
$$= \underline{\underline{\operatorname{sinc}\left(\frac{4B_0}{3}t\right) + \frac{2}{3}\operatorname{sinc}\left(\frac{8B_0}{3}t\right)}}$$

The second waveform is given by the inverse of the "triangular" spectrum. There exists several ways to reach the solution, some faster than the others.

i) Use the standard integral (this is rather straightforward but requires some calculations)

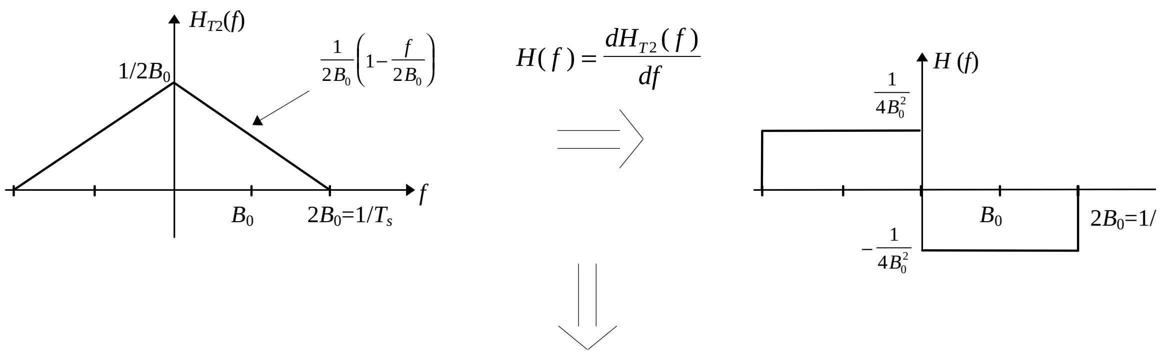
$$h_{T2}(t) = \int_{-\infty}^{\infty} H_{T2}(f) e^{j2\pi ft} df = \{\text{Spectrum is symmetric}\} = 2\int_{0}^{2B_0} \frac{1}{2B_0}\left(1 - \frac{f}{2B_0}\right)\cos(2\pi ft) df$$
$$= \{\text{Integration by parts}\} = \frac{1}{B_0}\left(1 - \frac{f}{2B_0}\right)\frac{\sin(2\pi ft)}{2\pi t}\Bigg|_{f=0}^{2B_0} + \int_{0}^{2B_0} \frac{1}{2B_0^2}\frac{\sin(2\pi ft)}{2\pi t} df$$
$$= 0 + \frac{1}{2}\frac{\cos(2\pi ft)}{(2\pi B_0 t)^2}\Bigg|_{f=0}^{2B_0} = \frac{1}{(2\pi B_0 t)^2}\frac{1 - \cos(4\pi B_0 t)}{2} = \frac{1}{(2\pi B_0 t)^2}\sin^2(2\pi B_0 t)$$
$$= \underline{\underline{\operatorname{sinc}^2(2B_0 t)}}$$

ii) Use the convolution property. The triangular spectrum given by $H_{T2}(f)$ can be obtained by convoluting two "rectangular" spectra both with a bandwidth of B_0. We can now use the convolution property (convolution in frequency-domain corresponds to multiplication in time-domain). The figure below illustrates the idea.



$$h_{T2}(t) = F^{-1}\{H_{T2}(f)\} = F^{-1}\{H_1(f) * H_2(f)\} = h_1(t)\cdot h_2(t) = \underline{\underline{\operatorname{sinc}^2(2B_0 t)}}$$

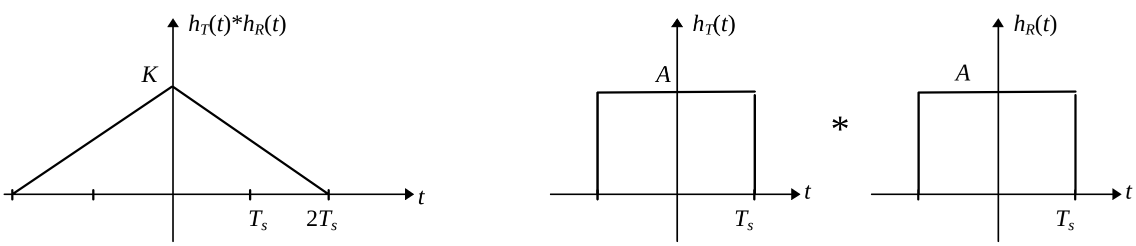
iii) We can also use the property of differentiating in frequency domain.



$$-j2\pi t \cdot h_{T2}(t) = F^{-1}\left\{\frac{dH_{T2}(f)}{df}\right\} \Rightarrow h_{T2}(t) = \underline{\underline{\mathrm{sinc}(2B_0 t)}}$$

Solution E 1-2

This type of problem is in general easier to solve in frequency domain by first transforming the time-domain pulse into frequency-domain and then split the spectrum between the transmitter and the receiver (see lecture notes). However, here we can use the results obtained in exercise 1-1 b. We know that a "triangle" shape is obtained by convoluting two rectangular pulses so the answer follows directly



We now only need to find the amplitude A in terms of K. This can be found by evaluating the convolution integral when the pulses overlap.

$$K = \int_{-T_s}^{T_s} A^2 dt = 2A^2 T_s \quad \Leftrightarrow \quad \underline{\underline{A = \pm\sqrt{\frac{K}{2T_s}}}}$$

In order to get a causal system we need also to insert a delay (see lecture notes).