Switch Fabrics

Switching Technology S38.165
http://www.netlab.hut.fi/opetus/s38165

Switch fabrics

- Multipoint switching
- Self-routing networks
- Sorting networks
- **Fabric implementation technologies**
- Fault tolerance and reliability

Fabric implementation technologies

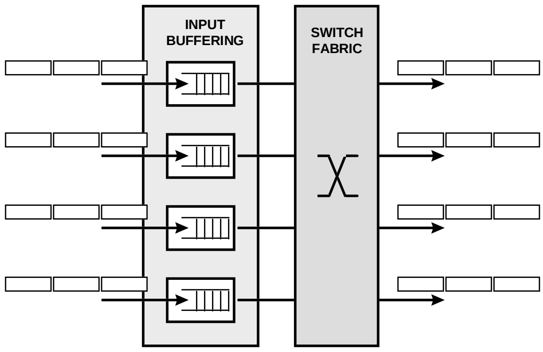
- Time division fabrics
  - Shared media
  - Shared memory
- Space division fabrics
  - Crossbar
  - Multi-stage constructions
- **Buffering techniques**

Buffering alternatives

- Input buffering
- Output buffering
- Central buffering
- Combinations
  – input-output buffering
  – central-output buffering

Input buffering

Buffer memories at the input interfaces

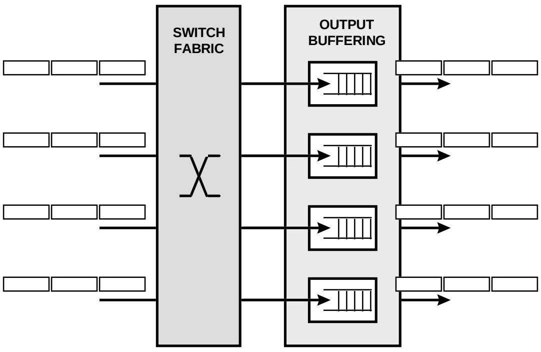


Input buffering (cont.)

- Pros
  - required memory access speed
    - in FIFO and dual-port RAM solutions equal to incoming line rate
    - in one-port RAM solutions *twice* the incoming line rate
  - Speed of switch fabric
    - multi-stages and crossbars operate at input wire speed
    - shared media fabrics operate at the aggregate speed of inputs
  - low cost solution (due to low memory speed)
- Cons
  - FIFO type of buffering => HOL problem
  - buffer size may be large (due to HOL)
  - HOL avoided by having a buffer for each output at each input

Output buffering

Buffer memories at the output interfaces

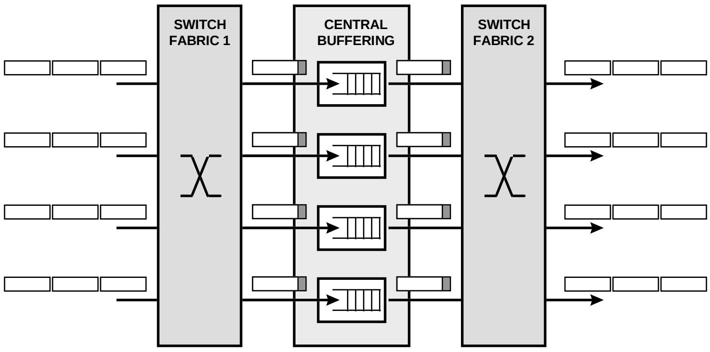


Output buffering (cont.)

- Pros
  - better throughput/delay performance than in input buffered systems
  - no HOL problem
- Cons
  - access speed of buffer memory
    - in FIFO and dual-port RAM solutions *N* times the incoming line rate
    - in one-port RAM solutions *N*+1 times the incoming line rate
  - high cost due to high memory speed requirement
  - switch fabric operates at the aggregate speed of inputs (*N* x wire speed)

Central buffering

Buffer memory located between two switch fabrics
- shared by all inputs/outputs
- virtual buffer for each input or output

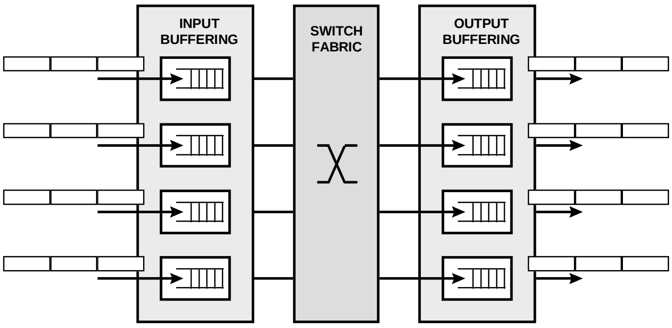


Central buffering (cont.)

- Pros
  - smaller buffer size requirement and lower average delay than in input or output buffering
  - HOL problem can be avoided
- Cons
  - speed of buffer memory
    - in dual-port RAM solutions larger than *N* times the incoming line rate
    - in one-port RAM solutions larger than 2x*N* times the incoming line rate
  - speed of switch fabric *N* x wire speed
  - complicated buffer control
  - high cost due to high memory speed requirement and control complexity

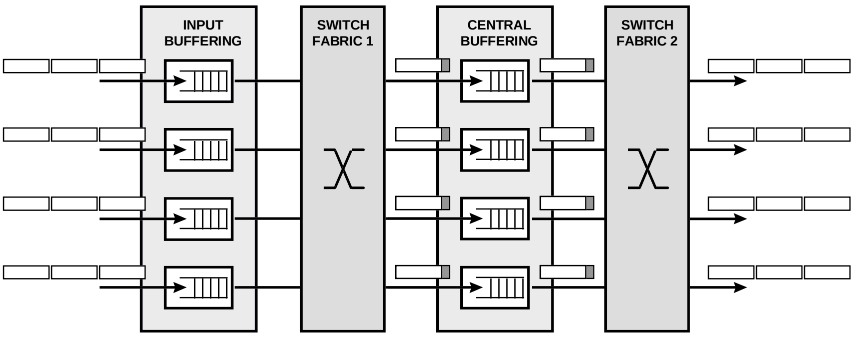
Input-output buffering

Input-output buffering common in QoS aware switches/routers
- inputs implement output specific buffers to avoid HOL
- outputs implement dedicated buffers for different traffic classes
- combined buffering distributes buffering complexity between inputs and outputs



Input-central buffering

Input-central buffering used in QoS aware switches/routers
- inputs implement output specific buffers to avoid HOL
- central buffer implements dedicated buffers for different traffic classes for each output

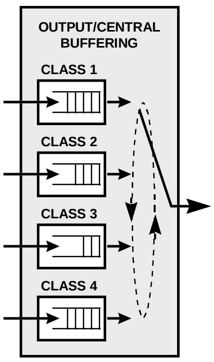


Summary of buffering techniques

Buffering principle	Memory space	Memory speed	Memory control	Queueing delay	Multi-casting capabilities
Input buffering	high	slow (~input rate)	simple	longest (due to HOL)	extra logic needed
Output buffering	medium	fast (~N x input rate)	simple	medium	supported
Central buffering	low	fast (~N x input rate)	complicated	shortest	supported but complex

Priorities and buffering

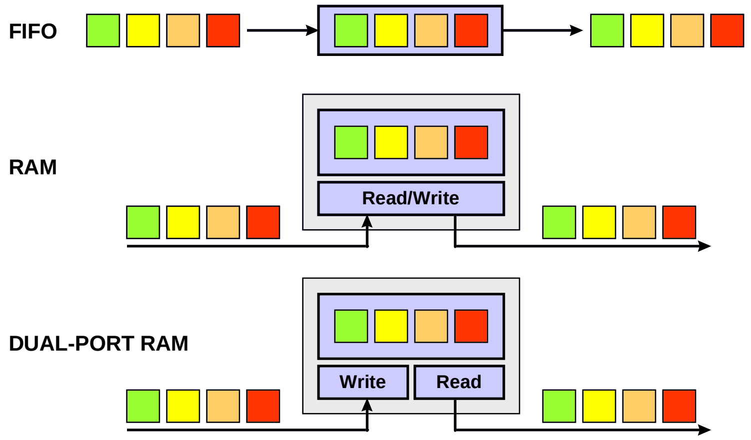
- Separate buffer for each traffic class
- A scheduler needed to control transmission data
  - highest priority served first
  - longest queue served first
  - minimization of lost packets/cells
- Priority given to high quality traffic
  - low delay and delay variation traffic
  - low loss rate traffic
  - best customer traffic
- Scheduling principles
  - round robin
  - weighted round robin
  - fair queuing
  - weighted fair queuing
  - etc.



Basic memory types for buffering

- FIFO (First-In-First-Out)
- RAM (Random Access Memory)
- Dual-port RAM

Basic memory types for buffering (cont.)



Switch fabrics

- Multipoint switching
- Self-routing networks
- Sorting networks
- Fabric implementation technologies
- **Fault tolerance and reliability**

Fault tolerance and reliability

- Definitions
- Fault tolerance of switching systems
- Modeling of tolerance and reliability

Definitions

- **Failure, malfunction -** is deviation from the intended/specified performance of a system
- **Fault -** is such a state of a device or a program which can lead to a failure
- **Error -** is an incorrect response of a program or module. An error is a indication that the module in question may be faulty, the module has received wrong input or it has been misused. An error can lead to a failure if the system is not tolerant to this sort of an error. A fault can exist without any error taking place.

Fault tolerance

- **Fault tolerance** is the ability of a system to continue its intended performance in spite of a fault or faults
- **A switching system** is an example of a fault tolerant system
- Fault tolerance always requires redundancy of some sort

Categorization of faults

- **Duration based**
  - **permanent** or stuck-at (stuck at zero or stuck at one)
  - **intermittent** - fault requires repair actions, but its impact is not always observable
  - **transient** - fault can be observed for a short period of time and disappears without repair
- **Observable** or **latent** (hidden)
- Based on the **scope** of the impact (serious - less serious)

Graceful degradation

- **Capability of a system to continue its functions under one or more faults, but on a reduced level of performance**
- **For example**
  - in some RAID (Redundant Array Inexpensive Disks) configurations, write speed drops in case of a disk fault, but continues on a lower level of performance even while the fault has not been repaired

Reliability and availability

- **Reliability *R(t)* -** probability that a system does not fail within time ***t*** under the condition that it was functioning correctly at ***t*** = 0
  - for all known man-made systems ***R(t)*** $\rightarrow$ **0** when ***t*** $\rightarrow \infty$
- **Availability *A(t)* -** probability that a system will function correctly at time *t*
  - for a system that can be repaired ***A(t)*** approaches some value asymptotically during the useful lifetime of the system

Repairable system

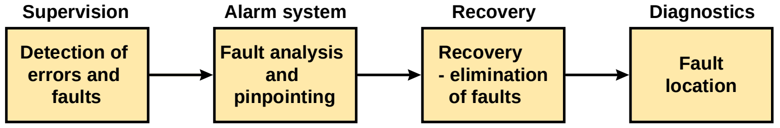
- **Maintainability *M(t)* -** probability that a system is returned to its correct functioning state during time *t* under the condition that it was faulty at time ***t*** = 0

MTTF, MTTR and MTBF

- **MTTF (Mean-Time-To-Failure) -** expected value of the time duration from the present to the next failure
- **MTTR (Mean-Time-To-Repair) -** expected value of the time duration from a fault until the system has been restored into a correct functioning state
- **MTBF (Mean-Time-Between-Failures) -** expected value of the time duration from occurrence of a fault until the next occurrence of a fault
  - **MTBF = MTTR + MTTR**

High availability of a switching system

- **High availability of a switching system is obtained by maintenance software**



Supervision
- In a unit under normal working load
- HW implementation => fast
- SW implementation => detection delay

Alarm system
- Often a rule based system

Recovery

Utilizes
- redundancy
- switch-overs
  - active <=> standby
- restarts
  - a single program
  - a preprocessor
  - a single main processor
  - whole system
  - fall back to previous SW package

Diagnostics
- In a unit temporarily without normal load

Maintenance software is one of the most important software sub-systems in a switching system in parallel with call/connection control and charging

Main types of redundancy

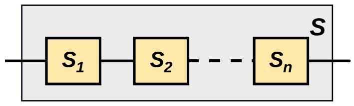
- **Hardware redundancy**
  - duplication (1+1) - need for “self-checking”-recovery blocks that detect their own faults
  - *n+r* -principle (*n* active units and *r* standby units)
- **Software redundancy**
  - required always in telecom systems
- **Information redundancy**
  - parity bits, block codes, etc.
- **Time redundancy**
  - delayed re-execution of transactions

Modeling of reliability

- Combinatorial models
- Markov analysis
- Other modeling techniques (not covered here)
  - Fault tree analysis
  - Reliability block diagrams
  - Monte Carlo simulation

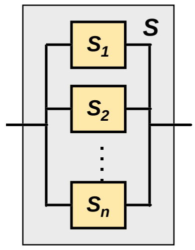
Combinatorial reliability

- A **serial system *S*** functions if and only if all its parts S_i ($1 \le i \le n$) function



=> $R_s = \prod_{i=1}^{n} R_i$ and $F_s = (1 - R_s)$

- Failures in sub-systems are supposed to be independent
- A **parallel** (replicated) **system** fails if all its sub-systems fail

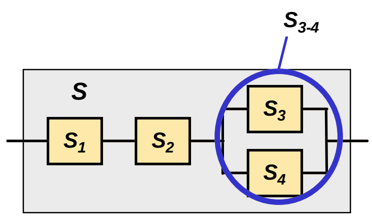


=> $F_s = \prod_{i=1}^{n} (1-R_i)$ and $R_s = 1 - F_s = 1 - \prod_{i=1}^{n} (1-R_i)$

- Reliability of a duplicated system ($R_i = R$) is $R_s = 1 - (1-R)^2$

Combinatorial reliability example 1

- Calculate reliability R_s and failure probability F_s of system S given that failures in sub-systems S_i are independent and for some time interval it holds that
$R_1 = 0.90$, $R_2 = 0.95$ and $R_3 = R_4 = 0.80$



=> $R_s = \prod R_i = R_1 \times R_2 \times R_{3\text{-}4}$

=> $R_{3\text{-}4} = 1 - \prod (1-R_i) = 1 - (1 - R_3)(1 - R_4)$

=> $R_s = R_1 \times R_2 \times [1 - (1 - R_3)(1 - R_4)]$

=> $F_s = 1 - R_s = 1 - R_1 \times R_2 \times [1 - (1 - R_3)(1 - R_4)]$

=> $R_s = 0.82$ and $F_s = 0.18$

Combinatorial reliability (cont.)

- A load sharing system functions if *m* of the total of *n* sub-systems function
- If failures in sub-systems S_i are independent then probability that the system fails is

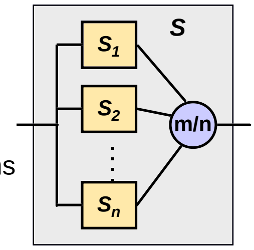
P(fails) = P(*k*<*m*)

and probability that it functions is

P(functioning) = P(*k*≥*m*) = 1- P(*k*<*m*)

where *k* is the number of functioning sub-systems

$$P(k \geq m) = \sum_{i=m}^{n} P(k=i) \quad \text{and} \quad P(k<m) = \sum_{i=0}^{m-1} P(k=i)$$



Combinatorial reliability example 2

- As an example, suppose we have a system having *m*=2 and *n*=4 and each of the four sub-systems have a different *R*, i.e. R_1, R_2, R_3 and R_4, and failures in sub-systems S_i are independent
- Probability that the system fails is

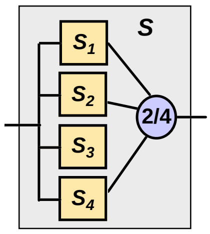
$$P(\text{fails}) = P(k<2) = \sum_{i=0}^{1} P(k=i) = P(k=0) + P(k=1)$$

- P(k=0) and P(k=1) can be derived to be

$$P(k=0) = (1- R_1)(1- R_2)(1- R_3)(1- R_4)$$

$$P(k=1) = R_1(1- R_2)(1- R_3)(1- R_4) + (1- R_1)R_2(1- R_3)(1- R_4) + (1- R_1)(1- R_2) R_3(1- R_4) + (1- R_1) (1- R_2)(1- R_3) R_4$$

- If R_1=0.9 ,R_2,=0.95 ,R_3 =0.85 and R_4 =0.8 then R_s = 0.994 and F_s = 0.0058



Combinatorial reliability (cont.)

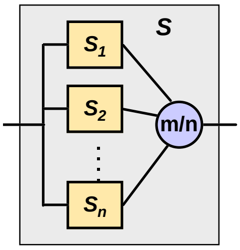
- If failures in sub-systems S_i of an m/n system are independent and $R_i = R$ for all $i \in [1,n]$ then the system is a Bernoulli system and binomial distribution applies

  $$\Rightarrow R_s = \sum_{k=m}^{n} \binom{n}{k} R^k (1-R)^{n-k}$$

- For a system of $m/n = 2/3$

  $$\Rightarrow R_{2/3} = \sum_{k=2}^{3} \frac{3!}{k!(3-k)!} R^k (1-R)^{3-k} = 3R^2 - 2R^3$$

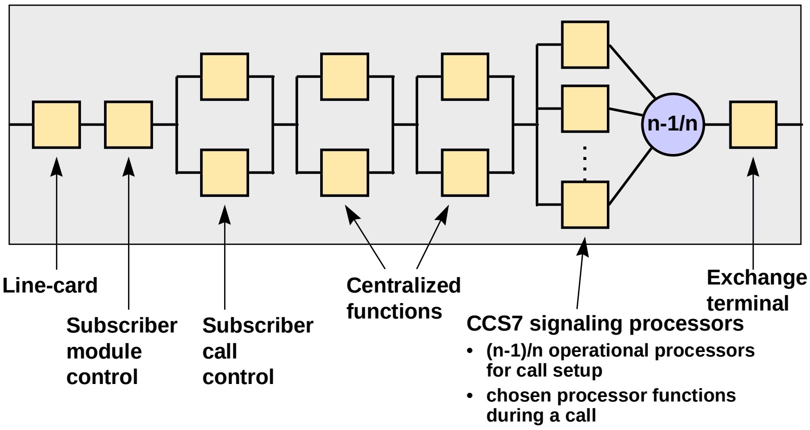
  If for example $R = 0.9 \Rightarrow R_{2/3} = 0.972$



Computing MTTF

- **MTTF =** $\int_0^\infty$ **R(t)dt -** valid for any reliability distribution
- Single component with a constant failure rate (CFR) λ
  - **$R(t) = e^{-\lambda t}$**
  - **$MTTF = 1/\lambda$**
- Serial systems with n CFR components
  - **$R_s(t) = R_1(t) \times R_2(t) \times \ldots \times R_n(t) = e^{-(\lambda_1 + \lambda_2 + \ldots + \lambda_n)t} = e^{-\lambda_s t}$**
  - **$\lambda_s = \lambda_1 + \lambda_2 + \ldots + \lambda_n$**
- **$MTTF_s = 1/\lambda_s$**
- **$1/MTTF_s = 1/MTTF_1 + 1/MTTF_2 + \ldots + 1/MTTF_n$**

Telecom exchange reliability from subscriber's point of view



Premature release requirement **P** ≤ **2x10^{-5}** applied

Failure intensity

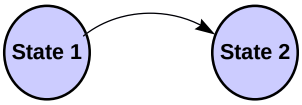
- Unit of failure intensity λ is defined to be
  [λ] = fit = number of faults /10^9 h
- Failure intensities for replaceable plug-in-units varies in the range 0.1 - 10 kfit
- Example:
  - if failure intensity of a line-card in an exchange is 2 kfit, what is its MTTF ?

$$\textbf{MTTF} = \mathbf{1/\lambda} = \frac{10^9\ \mathbf{h}}{2000} = \frac{1\ 000\ 000\ \mathbf{h}}{2x24x360} = 58 \text{ years}$$

Reliability modeling using Markov chains

Markov chains

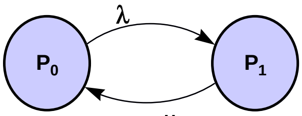
- A system is modeled as a set of states of transitions
- Each state corresponds to fulfillment of a set of conditions and each transition corresponds to an event in a system that changes from one state to another



- By using this method it is possible to find reliability behavior of a complex system having a number of states and non-independent failure modes

Markov chain modeling

- A set of states of transitions leads to a group of linear differential equations
- For a given modeling goal it is essential to choose a minimal set of states for equations to be easily solved
- By setting the derivatives of the probabilities to zero an asymptotic state is obtained if such exists



λ = failure intensity

μ = repair intensity (repair time is exponentially distributed)

P_i = probability of state i, e.g. $P_0 = R(t)$ and $P_1 = F(t)$,

Markov chain modeling (cont.)

- Probabilities (π_i) of the states and transition rates (λ_{ij}) between the states are tied together with the following formula

$$\pi\Lambda = \mathbf{0}$$

where

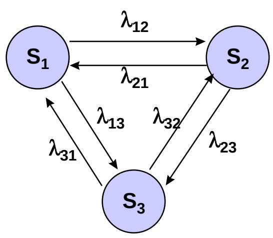
$$\pi = [\pi_1 \quad \pi_2 \quad \dots \quad \pi_n]$$

$$\Lambda = \begin{bmatrix} -(\lambda_{12}+\lambda_{13}+\cdots) & \lambda_{12} & \lambda_{13} & \cdots \\ \lambda_{21} & -(\lambda_{21}+\lambda_{23}+\cdots) & \lambda_{23} & \cdots \\ \lambda_{31} & \lambda_{32} & -(\lambda_{31}+\lambda_{32}+\cdots) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Markov chain modeling (cont.)

Example

$$\Lambda = \begin{bmatrix} -(\lambda_{12}+\lambda_{13}) & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & -(\lambda_{21}+\lambda_{23}) & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & -(\lambda_{31}+\lambda_{32}) \end{bmatrix}$$

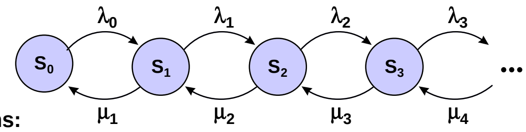


$\pi\Lambda = \mathbf{0}$ and $\pi = [\pi_1 \quad \pi_2 \quad \dots \quad \pi_n]$

$$\begin{cases} -(\lambda_{12}+\lambda_{13})\pi_1 + \lambda_{12}\pi_2 + \lambda_{13}\pi_3 = \mathbf{0} \\ \lambda_{21}\pi_1 - (\lambda_{21}+\lambda_{23})\pi_2 + \lambda_{23}\pi_3 = \mathbf{0} \\ \lambda_{31}\pi_1 + \lambda_{32}\pi_2 - (\lambda_{31}+\lambda_{32})\pi_3 = \mathbf{0} \end{cases}$$

Birth-death process

Birth-death process is a special case of continuous-time Markov chain, which models the size of population that increases by 1 (birth) or decreases by one (death).



Balance equations:

- **State S_0** $\quad \lambda_0\pi_0 = \lambda_1\pi_1 \quad => \quad \pi_1 = \frac{\lambda_0}{\mu_1}\pi_0$

- **State S_1** $\quad (\lambda_1 + \mu_1)\pi_1 = \lambda_0\pi_0 + \lambda_2\pi_2 \quad => \quad \pi_2 = \frac{\lambda_1\lambda_0}{\mu_2\mu_1}\pi_0$

- **State S_k** $\quad (\lambda_{k-1} + \mu_{k-1})\pi_{k-1} = \lambda_{k-2}\pi_{k-2} + \lambda_k\pi_k \quad => \quad \pi_k = \frac{\lambda_{k-1}\cdots\lambda_1\lambda_0}{\mu_k\cdots\mu_2\mu_1}\pi_0$

Birth-death process (cont.)

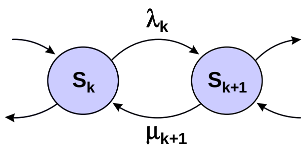
$$\pi_k = \left(\frac{\lambda_{k-1}}{\mu_k}\right)\cdots\left(\frac{\lambda_1}{\mu_2}\right)\left(\frac{\lambda_0}{\mu_1}\right)\pi_0 = \rho_{k-1}\cdots\rho_1\rho_1\pi_0 \quad \text{where} \quad \rho_k = \frac{\lambda_k}{\mu_{k+1}} \quad (k=1, 2, 3, \ldots)$$

Substituting these expressions for π_k into $\sum_{k=0}^{\infty}\pi_k = 1$ yields

$$\pi_0 + \sum_{k=1}^{\infty}\frac{\lambda_{k-1}\cdots\lambda_1\lambda_0}{\mu_k\cdots\mu_2\mu_1}\pi_0 = 1 \quad => \quad \pi_0\left[1 + \sum_{k=1}^{\infty}\frac{\lambda_{k-1}\cdots\lambda_1\lambda_0}{\mu_k\cdots\mu_2\mu_1}\right] = 1$$

$$=> \quad \frac{1}{\pi_0} = \left[1 + \sum_{k=1}^{\infty}\frac{\lambda_{k-1}\cdots\lambda_1\lambda_0}{\mu_k\cdots\mu_2\mu_1}\right]$$

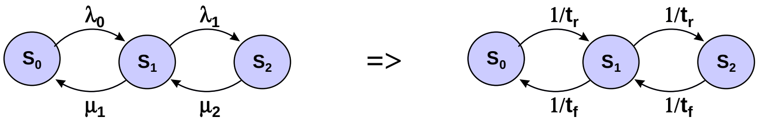
$$=> \quad \pi_k = \frac{\lambda_{k-1}\cdots\lambda_1\lambda_0}{\mu_k\cdots\mu_2\mu_1}\pi_0 \quad (k=1, 2, 3, \ldots)$$



Example of birth-death process

A switching system has two control computer, one on-line and one standby. The time interval between computer failures is exponentially distributed with mean t_f . In case of a failure, the standby computer replaces the failed one.
A single repair facility exist and repair times are exponentially distributed with mean t_r .
What fraction of time the system is out of use, i.e., both computers having failed?

The problem can be solved by using a three state birth-death model.



Example of birth-death process (cont.)

S_0 - both computer operable
S_1 - one computer failed
S_2 - both computer failed

$$\frac{1}{\pi_0} = \left[1 + \frac{1/t_r}{1/t_f} + \left(\frac{1/t_r}{1/t_f} \right)^2 \right] \quad => \quad \pi_0 = \frac{t_r^2}{t_r^2 + t_r t_f + t_f^2}$$

(probability that both computers have failed)

If t_r/t_f = 10 , i.e. the average repair time is 10 % of the average time between failures, then π_0 =0.009009 and both computer will be out of service 0.9 % of the time.

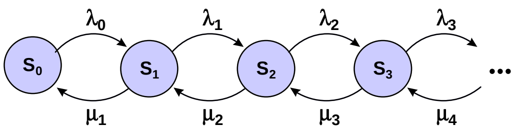
Additional reading of Markov chain modeling

Switching Technology S38.165
http://www.netlab.hut.fi/opetus/s38165

Markov chain modeling

A continuous-time Markov Chain is a stochastic process $\{X(t): t \geq 0\}$

- $X(t)$ can have values is S={0,1,2,3,...}
- Each time the process enters a state *i*, the amount of time it spends in that state before making a transition to another state has an exponential distribution with mean $1/\lambda_i$
- When leaving state *i*, the process moves to a state *j* with probability p_{ij} where p_{ii}=0
- The next state to be visited after *i* is independent of the length of time spend in state *i*



Markov chain modeling (cont.)

Transition probabilities

$$p_{ij}(t) = P\{X(t+s) = j \mid X(s) = i\}$$

Continuous at t=0, with

$$\lim_{t \to 0} p_{ij}(t) = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$$

Transition matrix is a function of time

$$P(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots \\ p_{21}(t) & \vdots & \\ \vdots & & \ddots \end{bmatrix}$$

Markov chain modeling (cont.)

Transition intensity:

$$\lambda_j(t) = -\frac{d}{dt} p_{jj}(0)$$ (rate at which the process leaves state *j* when it is in state *j*)

$$\lambda_{ij}(t) = \frac{d}{dt} p_{ij}(0) = \lambda_i p_{ij}$$ (transition rate into state *j* when the process in is state *i*)

The process, starting in state *i*, spends an amount of time in that state having exponential distribution with rate λ_i. It then moves to state *j* with probability

$$p_{ij} = \frac{\lambda_{ij}}{\lambda_i} \quad \forall i, j \qquad \sum_{j=1}^{n} p_{ij} = \sum_{j=1}^{n} \frac{\lambda_{ij}}{\lambda_i} = \frac{\sum_{j=1}^{n} \lambda_{ij}}{\lambda_i} = 1 \quad \Rightarrow \quad \lambda_i = \sum_{j=1}^{n} \lambda_{ij}$$

Markov chain modeling (cont.)

Chapman-Kolmogorov equations:

$$p_{ij}(t+s) = \sum_{k \in S} p_{ik}(t) p_{kj}(s) \qquad \forall i, j \in S \quad \forall s, t \geq 0$$

Since *p(t)* is a continuous function

$$p_{ij}(\Delta t) = p_{ij}(0) + \frac{d}{dt} p_{ij}(0)\Delta t + o(\Delta t^2)$$

We have defined => $\lambda_{ij}(t) = \frac{d}{dt} p_{ij}(0)$

For *i≠j*: $p_{ij}(\Delta t) = p_{ij}(0) + \lambda_{ij}\Delta t + o(\Delta t^2) \approx \lambda_{ij}\Delta t$ (for small Δt)

For *i=j*: $p_{ii}(\Delta t) = p_{ii}(0) + \lambda_{ii}\Delta t + o(\Delta t^2) \approx 1 + \lambda_{ii}\Delta t$ (for small Δt)

Markov chain modeling (cont.)

From Chapman-Kolmogorov equations:

$$p_{ij}(t+\Delta t) = \sum_k p_{ik}(t) p_{kj}(\Delta t) = p_{ij}(t) p_{jj}(\Delta t) + \sum_{k \neq j} p_{ik}(t) p_{kj}(\Delta t)$$

$$= p_{ij}(t)\left[1 + \lambda_{jj}\Delta t + o(\Delta t^2)\right] + \sum_{k \neq j} p_{ik}(t)\left[\lambda_{kj}\Delta t + o(\Delta t^2)\right]$$

$$p_{ij}(t+\Delta t) = p_{ij}(t) + \left[\sum_k p_{ik}(t)\lambda_{kj}\right]\Delta t + \left[\sum_k p_{ik}(t)\right] o(\Delta t^2)$$

$$\frac{p_{ij}(t+\Delta t) - p_{ij}(t)}{\Delta t} = \sum_k p_{ik}(t)\lambda_{kj} + \left[\sum_k p_{ik}(t)\right]\frac{o(\Delta t^2)}{\Delta t}$$

Taking the limit as $\Delta t \longrightarrow 0$

$$\boxed{\frac{d}{dt} p_{ij}(t) = \sum_k p_{ik}(t)\lambda_{kj} \qquad \forall i, j}$$

Markov chain modeling (cont.)

The process is described by the system of differential equations:

$$\frac{d}{dt} p_{ij}(t) = \sum_k p_{ik}(t)\lambda_{kj} \qquad \forall i,j$$

which can be given in the form

$$\frac{d}{dt} P(t) = P(t)\Lambda \qquad \forall i,j$$

$$\sum_j p_{ij}(t) = 1 \qquad \forall i,t$$

$$\frac{d}{dt}\sum_j p_{ij}(t) = \frac{d}{dt}(1) = 0$$

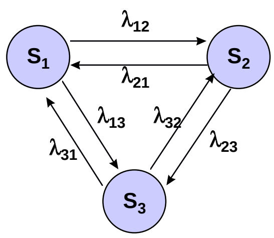
$$\frac{d}{dt}\sum_j p_{ij}(t) = 0$$

$$\sum_j \lambda_{ij} = 0$$

The sum of of each row of Λ is zero !

Markov chain modeling (cont.)

Example



$$\Lambda = \begin{bmatrix} -(\lambda_{12}+\lambda_{13}) & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & -(\lambda_{21}+\lambda_{23}) & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & -(\lambda_{31}+\lambda_{32}) \end{bmatrix}$$

The sum of of each row of Λ must be zero !

Markov chain modeling (cont.)

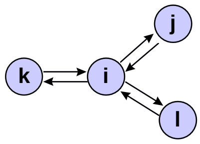
Steady state probabilities

$\lim_{t\to\infty} p_{ij}(t) = \pi_j$ **(Independent of initial state *i*)**

Must be non-negative and must satisfy $\sum_{i=1}^{n} \pi_i = 1$

In case of continuous-time Markov chains balance equation used to determine π.
For each state *i*, the rate at which the system leaves the state must equal to the rate at which the system enters the state

=> $\lambda_i \pi_i = \lambda_{ji}\pi_j + \lambda_{ki}\pi_k + \lambda_{li}\pi_l$



Markov chain modeling (cont.)

Balance equation

$$\left(\sum_{j\neq i} \lambda_{ij}\right)\pi_i = \sum_{k\neq i} \lambda_{ki}\pi_k \qquad \forall i$$

Steady state distribution is computed by solving this system of equations

$$\left(\sum_{j\neq i} \lambda_{ij}\right)\pi_i = \sum_{k\neq i} \lambda_{ki}\pi_k \qquad \forall i$$

$$\sum_{i=1}^{n} \pi_i = 1$$

Markov chain modeling (cont.)

An alternative derivation of the steady-state conditions begins with the differential equation describing the process:

$$\frac{d}{dt} p_{ij}(t) = \sum_k p_{ik}(t)\lambda_{kj} \qquad \forall i, j$$

Suppose that we take the limit of each side as $t \to \infty$

=> $\lim_{t\to\infty} \frac{d}{dt} p_{ij}(t) = \lim_{t\to\infty} \sum_k p_{ik}(t)\lambda_{kj}$

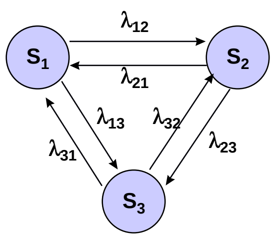
=> $\frac{d}{dt} \lim_{t\to\infty} p_{ij}(t) = \sum_k \lim_{t\to\infty} p_{ik}(t)\lambda_{kj}$

=> $\sum_k \pi_k \lambda_{kj} = 0$ **i.e. $\pi\Lambda=0$**

Markov chain modeling (cont.)

Example

$$\Lambda = \begin{bmatrix} -(\lambda_{12}+\lambda_{13}) & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & -(\lambda_{21}+\lambda_{23}) & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & -(\lambda_{31}+\lambda_{32}) \end{bmatrix}$$



$\pi\Lambda = 0$ and $\pi = [\pi_1 \quad \pi_2 \quad \dots \quad \pi_n]$

$$\begin{cases} -(\lambda_{12}+\lambda_{13})\pi_1 + \lambda_{21}\pi_2 + \lambda_{31}\pi_3 = 0 \\ \lambda_{12}\pi_1 - (\lambda_{21}+\lambda_{23})\pi_2 + \lambda_{32}\pi_3 = 0 \\ \lambda_{13}\pi_1 + \lambda_{23}\pi_2 - (\lambda_{31}+\lambda_{32})\pi_3 = 0 \end{cases}$$