



Batch means method

- Batch means method is used frequently
- Simulation is done as a single (long) run
 - let the length of simulation be M
 - * here we think that we consider the system from a customer point of view; then M may mean the number of interesting observations (as well we may think that M represents time)
 - let the observed variable be X (for instance, waiting time in a queue) and the task is to estimate its expected value $\mu = E[X]$
- From the beginning of the simulation, the warm-up period of K observations is rejected
- The useful run (of length $M - K$) is divided into N batches; thus in each batch there are

$$n = \frac{M - K}{N}$$

observations



Batch means method (continued)

- In batch i we get for X the sample average (X_{ij} denotes the j^{th} observation in the i^{th} batch)

$$\bar{X}_i = \frac{1}{n} \sum_{j=1}^n X_{ij}$$

- The final estimator for the expectation μ is

$$\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^N \bar{X}_i = \frac{1}{nN} \sum_{i=1}^N \sum_{j=1}^n X_{ij}$$

- This is simply the sample average of the whole run (after the warm-up period)
 - the division in batches has no bearing from the point of view of the estimator
 - the sole purpose of the division is to get an idea of the confidence interval of the estimator
- Assuming that the batches are long enough, the sample averages $\bar{X}_i$ of the batches are approximately independent
- Their sample variance then provides an estimate for the variance of a single $\bar{X}_i$

$$S^2 = \frac{1}{N-1} \sum_{i=1}^N (\bar{X}_i - \hat{\mu}_N)^2$$



Batch means method (continued)

- The confidence interval of the estimator (at confidence level $1 - \beta$) is

$$\hat{\mu}_N \pm z_{1-\beta/2} \frac{S}{\sqrt{N}}$$

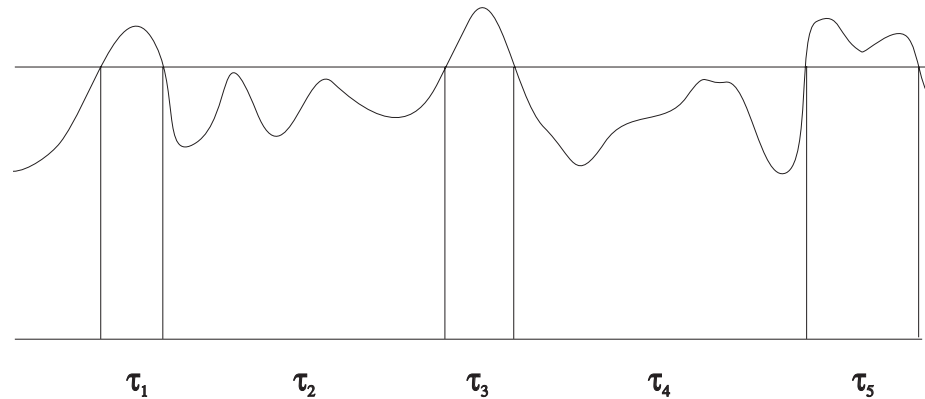
- The advantage of the method is that there is only one warm-up period
- There should be at least 20-30 batches in order to estimate the variance reliably
- The batches should be long enough (much longer than the duration of the initial transient) to guarantee that the $\bar{X}_i$ are approximately independent
- If there is dependence, the correlation is usually positive
- Then the real confidence interval of $\hat{\mu}$ is larger than the estimate given above based on the assumption of independent batches
 - the dependence does not at all degrade the value of the estimator
 - it only can mislead the user to believe that the accuracy of the estimator is better than it actually is

Regenerative method

- Is applicable in so called regenerative systems
- A regenerative system has at least one regenerative state
 - the stochastic development of the system from that point on does not at all depend on how this state has been reached
 - every state of a Markovian system is regenerative
 - in an $G/G/1$ queue the state where the system is empty is a regenerative state
- If there are several regenerative states, one of them is chosen as the basis for the data collection method
 - in the sequel, the regenerative state refers to the chosen regenerative state
- Every now and then the system visits the regenerative state or “regenerates itself”
 - this starts “a new life” which does not depend on the past

Regenerative method (continued)

- The instant, when the system returns to the regenerative state, is called the regeneration point
- The period between two regeneration points is called the regeneration period
- The developments of different regeneration periods are fully independent of each other
 - this is the “point” of the method



Regenerative method: point estimator

- Let X be the cumulative value of the observed variable during a regenerative period, for instance,
 - the total time the system has spent in a blocking state during a regenerative period
 - the total number of packets overflowed from a buffer during the period
- Let τ be the “duration” of the regenerative period
 - this may refer to the real duration (time) of the period
 - it may also refer to e.g. the total number of arrivals during the regenerative period
- The expectation of the observed variable ℓ (for instance, the expectation of time blocking) is

$$\ell = \frac{E[X]}{E[\tau]}$$

- In a simulation over n regenerative periods one obtains a (strongly consistent) estimator

$$\bar{\ell}_n = \frac{\bar{X}}{\bar{\tau}}$$

where $\bar{X}$ and $\bar{\tau}$ are the sample averages $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ and $\bar{\tau} = \frac{1}{n} \sum_{i=1}^n \tau_i$

The confidence interval of the estimator

- Consider the variable $Z_i = X_i - \ell\tau_i$
 - the Z_i are independent and identically distributed random variables (with mean 0)
 - so are the X_i and the τ_i

- Denote

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i, \quad \bar{\tau} = \frac{1}{n} \sum_{i=1}^n \tau_i, \quad \bar{Z} = \frac{1}{n} \sum_{i=1}^n Z_i = \bar{X} - \ell\bar{\tau}$$

- By the central limit theorem we have

$$\frac{n^{1/2}\bar{Z}}{\sigma} = \frac{n^{1/2}(\bar{X} - \ell\bar{\tau})}{\sigma} \rightarrow N(0, 1), \quad \text{kun } n \rightarrow \infty$$

where σ^2 is the variance of Z

$$\sigma^2 = V[Z] = V[X] - 2\ell\text{Cov}[X, \tau] + \ell^2 V[\tau]$$

The confidence interval of the estimator (continued)

- By dividing by $\bar{\tau}$ we get

$$\frac{n^{1/2}(\bar{\ell}_n - \ell)}{\sigma/\bar{\tau}} \rightarrow N(0, 1), \quad \text{when } n \rightarrow \infty$$

- For the point estimator $\bar{\ell}_n$ based on measurement over n regenerative periods we get the confidence interval (at the confidence level $1 - \beta$)

$$\bar{\ell}_n \pm \frac{z_{1-\beta/2} S}{\sqrt{n\bar{\tau}}}$$

where S^2 is the (unbiased) estimator of σ^2 based on the sample

$$S^2 = S_{11} - 2\bar{\ell}_n S_{12} + \bar{\ell}_n^2 S_{22}$$

and S_{11} , S_{22} and S_{12} are the sample variances and sample covariance of X and τ

$$S_{11} = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2, \quad S_{22} = \frac{1}{n-1} \sum_{i=1}^n (\tau_i - \bar{\tau})^2, \quad S_{12} = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})(\tau_i - \bar{\tau})$$



Regenerative method: discussion

- Advantages
 - separate transient removal is not needed
 - one does not have to fix parameters such as the number of batches in advance
 - asymptotically accurate
 - easy to understand and implement
- There are, however, a few disadvantages
 - it may be difficult to identify regenerative states
 - even if one can be identified
 - * the regenerative period may be very long (the user has no control over it)
 - * in a complex system the identification of the regenerative state may be computationally expensive
 - with a finite value of n the estimator $\bar{\ell}_n$ is biased
 - * in fact, the initial transient problem does exist, though it is somewhat concealed