6. Stochastic processes (2)

Contents

Markov process

- Consider a continuous-time and discrete-state stochastic process $X(t)$
  - with state space $S = \{0,1,\ldots,N\}$ or $S = \{0,1,...\}$
- **Definition**: The process $X(t)$ is a **Markov process** if

$$P\{X(t_{n+1}) = x_{n+1} \mid X(t_1) = x_1,\ldots,X(t_n) = x_n\} =$$
$$P\{X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n\}$$

for all n, $t_1 < \ldots < t_{n+1}$ and $x_1,\ldots, x_{n+1}$

- This is called the **Markov property**
  - Given the current state, the future of the process does not depend on its past (that is, *how* the process has evolved to the current state)
  - As regards the future of the process, the current state contains all the required information

Example

- Process $X(t)$ with independent increments is always a Markov process:

$$X(t_n) = X(t_{n-1}) + (X(t_n) - X(t_{n-1}))$$

- **Consequence:** Poisson process $A(t)$ is a Markov process:
  - according to Definition 3, the increments of a Poisson process are independent

Time-homogeneity

- **Definition**: Markov process $X(t)$ is **time-homogeneous** if

$$P\{X(t+\Delta)=y \mid X(t)=x\} = P\{X(\Delta)=y \mid X(0)=x\}$$

for all $t,\ \Delta \geq 0$ and $x, y \in S$

- In other words, probabilities $P\{X(t+\Delta)=y \mid X(t)=x\}$ are independent of t

State transition rates

- Consider a time-homogeneous Markov process $X(t)$
- The **state transition rates** q_{ij}, where $i, j \in S$, are defined as follows:

$$q_{ij} := \lim_{h \downarrow 0} \frac{1}{h} P\{X(h) = j \mid X(0) = i\}$$

- The initial distribution $P\{X(0) = i\}$, $i \in S$, and the state transition rates q_{ij} together determine the state probabilities $P\{X(t) = i\}$, $i \in S$, by the Kolmogorov (backwards/forwards) equations
- Note that on this course we will consider only time-homogeneous Markov processes

Exponential holding times

- Assume that a Markov process is in state i
- During a short time interval $(t, t+h]$, the conditional probability that there is a transition from state i to state j is $q_{ij}h + o(h)$ (independently of the other time intervals)
- Let q_i denote the total transition rate out of state i, that is:

$$q_i := \sum_{j \neq i} q_{ij}$$

- Then, during a short time interval $(t, t+h]$, the conditional probability that there is a transition from state i to any other state is $q_i h + o(h)$ (independently of the other time intervals)
- This is clearly a memoryless property
- Thus, the holding time in (any) state i is exponentially distributed with intensity q_i

State transition probabilities

- Let T_i denote the holding time in state i and T_{ij} denote the (potential) holding time in state i that ends to a transition to state j

$$T_i \sim \mathrm{Exp}(q_i), \quad T_{ij} \sim \mathrm{Exp}(q_{ij})$$

- T_i can be seen as the minimum of independent and exponentially

$$T_i = \min_{j \neq i} T_{ij}$$

- Let then p_{ij} denote the conditional probability that, when in state i, there is a transition from state i to state j (the **state transition probabilities**);

$$p_{ij} = P\{T_i = T_{ij}\} = \frac{q_{ij}}{q_i}$$

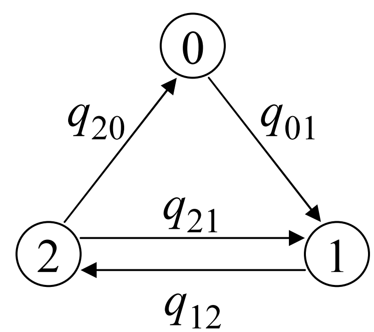
State transition diagram

- A time-homogeneous Markov process can be represented by a **state transition diagram**, which is a directed graph where
  - nodes correspond to states and
  - one-way links correspond to potential state transitions

$$\text{link from state } i \text{ to state } j \quad \Leftrightarrow \quad q_{ij} > 0$$

- Example: Markov process with three states, $S = \{0,1,2\}$

$$Q = \begin{pmatrix} - & + & 0 \\ 0 & - & + \\ + & + & - \end{pmatrix}$$



Irreducibility

- **Definition**: There is a **path** from state i to state j ($i \to j$) if there is a directed path from state i to state j in the state transition diagram.
  - In this case, starting from state i, the process visits state j with positive probability (sometimes in the future)
- **Definition**: States i and j **communicate** ($i \leftrightarrow j$) if $i \to j$ and $j \to i$.
- **Definition**: Markov process is **irreducible** if all states $i \in S$ communicate with each other
  - Example: The Markov process presented in the previous slide is irreducible

Global balance equations and equilibrium distributions

- Consider an irreducible Markov process $X(t)$, with state transition rates q_{ij}
- **Definition**: Let $\pi = (\pi_i \mid \pi_i \geq 0, i \in S)$ be a distribution defined on the state space S, that is:

$$\sum_{i \in S} \pi_i = 1 \qquad \text{(N)}$$

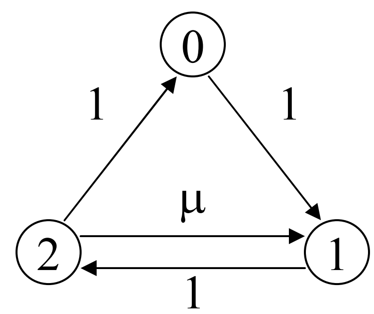
It is the **equilibrium distribution** of the process if the following **global balance equations** (GBE) are satisfied for each $i \in S$:

$$\sum_{j \neq i} \pi_i q_{ij} = \sum_{j \neq i} \pi_j q_{ji} \qquad \text{(GBE)}$$

  - It is possible that no equilibrium distribution exists, but if the state space is finite, a unique equilibrium distribution does exist
  - By choosing the equilibrium distribution (if it exists) as the initial distribution, the Markov process $X(t)$ becomes stationary (with stationary distribution π)

Example

$$Q = \begin{pmatrix} - & 1 & 0 \\ 0 & - & 1 \\ 1 & \mu & - \end{pmatrix}$$



$$\pi_0 + \pi_1 + \pi_2 = 1 \qquad \text{(N)}$$

$$\begin{aligned} \pi_0 \cdot 1 &= \pi_2 \cdot 1 \\ \pi_1 \cdot 1 &= \pi_0 \cdot 1 + \pi_2 \cdot \mu \\ \pi_2 \cdot (1+\mu) &= \pi_1 \cdot 1 \end{aligned} \qquad \text{(GBE)}$$

$$\Rightarrow \quad \pi_0 = \frac{1}{3+\mu}, \quad \pi_1 = \frac{1+\mu}{3+\mu}, \quad \pi_2 = \frac{1}{3+\mu}$$

Local balance equations

- Consider still an irreducible Markov process $X(t)$.with state transition rates q_{ij}
- **Proposition**: Let $\pi = (\pi_i \mid \pi_i \geq 0,\ i \in S)$ be a distribution defined on the state space S, that is:

$$\sum_{i \in S} \pi_i = 1 \qquad \text{(N)}$$

If the following **local balance equations** (LBE) are satisfied for each $i,j \in S$:

$$\pi_i q_{ij} = \pi_j q_{ji} \qquad \text{(LBE)}$$

then π is the equilibrium distribution of the process.

- **Proof**: (GBE) follows from (LBE) by summing over all $j \neq i$
- In this case the Markov process $X(t)$ is called **reversible** (looking stochastically the same in either direction of time)

Contents

Birth-death process

- Consider a continuous-time and discrete-state Markov process $X(t)$
  - with state space $S = \{0,1,\ldots,N\}$ or $S = \{0,1,...\}$
- **Definition**: The process $X(t)$ is a **birth-death process** (BD) if state transitions are possible only between neighbouring states, that is:

$$|i-j|>1 \quad \Rightarrow \quad q_{ij}=0$$

- In this case, we denote

$$\mu_i := q_{i,i-1} \geq 0$$

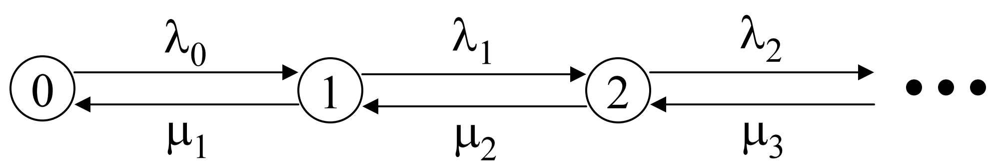
$$\lambda_i := q_{i,i+1} \geq 0$$

  - In particular, we define $\mu_0 = 0$ and $\lambda_N = 0$ (if $N < \infty$)

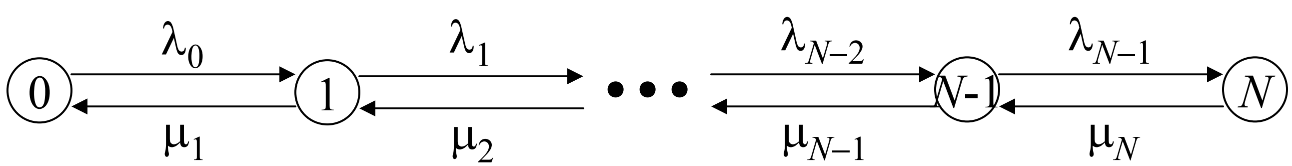
Irreducibility

- **Proposition**: A birth-death process is irreducible if and only if $\lambda_i > 0$ for all $i \in S\backslash\{N\}$ and $\mu_i > 0$ for all $i \in S\backslash\{0\}$

- State transition diagram of an infinite-state irreducible BD process:



- State transition diagram of a finite-state irreducible BD process:



Equilibrium distribution (1)

- Consider an irreducible birth-death process $X(t)$
- We aim is to derive the equilibrium distribution $\pi = (\pi_i \mid i \in S)$ (if it exists)
- Local balance equations (LBE):

$$\pi_i \lambda_i = \pi_{i+1} \mu_{i+1} \qquad \text{(LBE)}$$

- Thus we get the following recursive formula:

$$\pi_{i+1} = \frac{\lambda_i}{\mu_{i+1}} \pi_i \quad \Rightarrow \quad \pi_i = \pi_0 \prod_{j=1}^{i} \frac{\lambda_{j-1}}{\mu_j}$$

- Normalizing condition (N):

$$\sum_{i \in S} \pi_i = \pi_0 \sum_{i \in S} \prod_{j=1}^{i} \frac{\lambda_{j-1}}{\mu_j} = 1 \qquad \text{(N)}$$

Equilibrium distribution (2)

- Thus, the equilibrium distribution exists if and only if

$$\sum_{i \in S} \prod_{j=1}^{i} \frac{\lambda_{j-1}}{\mu_j} < \infty$$

- **Finite state space**:
  The sum above is always finite, and the equilibrium distribution is

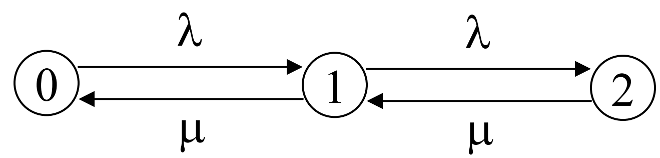
$$\pi_i = \pi_0 \prod_{j=1}^{i} \frac{\lambda_{j-1}}{\mu_j}, \quad \pi_0 = \left(1 + \sum_{i=1}^{N} \prod_{j=1}^{i} \frac{\lambda_{j-1}}{\mu_j} \right)^{-1}$$

- **Infinite state space**:
  If the sum above is finite, the equilibrium distribution is

$$\pi_i = \pi_0 \prod_{j=1}^{i} \frac{\lambda_{j-1}}{\mu_j}, \quad \pi_0 = \left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} \frac{\lambda_{j-1}}{\mu_j} \right)^{-1}$$

Example

$$Q = \begin{pmatrix} - & \lambda & 0 \\ \mu & - & \lambda \\ 0 & \mu & - \end{pmatrix}$$



$$\pi_i \lambda = \pi_{i+1} \mu$$

$$\Rightarrow \quad \pi_{i+1} = \rho \pi_i \quad (\rho := \lambda / \mu) \qquad \text{(LBE)}$$

$$\Rightarrow \quad \pi_i = \pi_0 \rho^i$$

$$\pi_0 + \pi_1 + \pi_2 = \pi_0 (1 + \rho + \rho^2) = 1 \qquad \text{(N)}$$

$$\Rightarrow \quad \pi_i = \frac{\rho^i}{1+\rho+\rho^2}$$

Pure birth process

- **Definition**: A birth-death process is a **pure birth process** if $\mu_i = 0$ for all $i \in S$
- State transition diagram of an infinite-state pure birth process:

$$\textcircled{0} \xrightarrow{\lambda_0} \textcircled{1} \xrightarrow{\lambda_1} \textcircled{2} \xrightarrow{\lambda_2} \bullet\bullet\bullet$$

- State transition diagram of a finite-state pure birth process:

$$\textcircled{0} \xrightarrow{\lambda_0} \textcircled{1} \xrightarrow{\lambda_1} \bullet\bullet\bullet \xrightarrow{\lambda_{N-2}} \textcircled{N\text{-}1} \xrightarrow{\lambda_{N-1}} \textcircled{N}$$

- Example: Poisson process is a pure birth process (with constant birth rate $\lambda_i = \lambda$ for all $i \in S = \{0,1,\ldots\}$)
- Note: Pure birth process is never irreducible (nor stationary)!