QUEUEING NETWORKS

A network consisting of several interconnected queues

- Network of queues

Examples

- Customers go form one queue to another in post office, bank, supermarket etc
- Data packets traverse a network moving from a queue in a router to the queue in another router

History

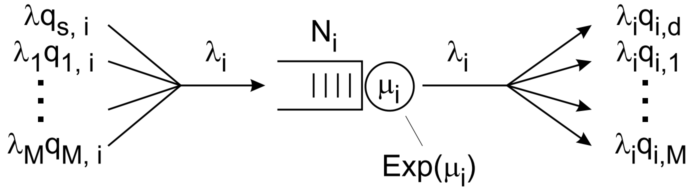
- Burke's theorem, Burke (1957), Reich (1957)
- Jackson (1957, 1963): open queueing networks, product form solution
- Gordon and Newell (1967): closed queueing networks
- Baskett, Chandy, Muntz, Palacios (1975): generalizations of the types of queues
- Reiser and Lavenberg (1980, 1982): mean value analysis, MVA

Jackson's queueing network (open queueing network)

Jackson's open queueing network consists of M nodes (queues) with the following assumptions:

- Node i is a FIFO queue
  - unlimited number of waiting places (infinite queue)
- Service time in the queue obeys the distribution Exp(μ_i)
  - in each queue, the service time of the customer is drawn independent of the service times in other queues
  - note: in a packet network the sending time of a packet, in reality, is the same in all queues (or differs by a constant factor, the inverse of the line speed)
  - this dependence, however, does not markedly affect the behaviour of the system (so called Kleinrock's independence assumption)
- Upon departure from queue i, the customer chooses the next queue j randomly with the probability $q_{i,j}$ or exits the network with the probability $q_{i,d}$ (probabilistic routing)
  - the model can be extended to cover the case of predetermined routes (route pinning)
- The network is open to arrivals from outside of the network (source)
  - from the source s customers arrive as a Poisson stream with intensity λ
  - fraction $q_{s,i}$ of them enter queue i (intensity $\lambda q_{s,i}$)

Node i in Jackson's network

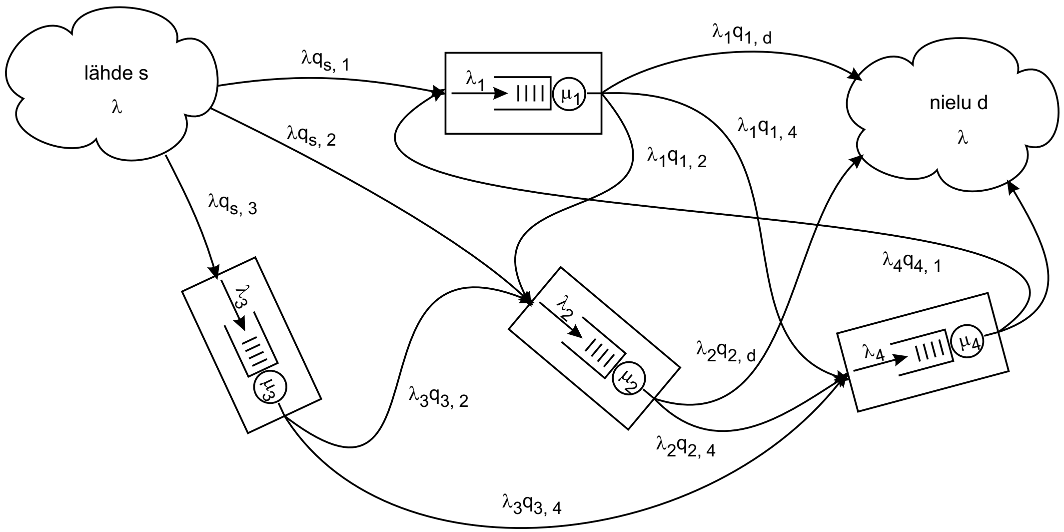


$$\begin{cases} s &= \text{source, external} \\ d &= \text{destination, sink} \\ N_i &= \text{number of customers in queue } i \end{cases}$$

Without complications one could assume state dependent service rates $\mu_i = \mu_i(N_i)$. This could describe e.g. multiserver nodes. To simplify the notation, we assume in the sequel a constant sevice rate μ_i.

Jackson's network

The opnenness of the network requires that from each node there is at leas one path ($\neq 0$) to the sink d, i.e. the probability that a customer entering the network will ultimately exit the network is 1.



Conservation of the flows

Denote λ_i = average customer flow through node i.

- Although the external arrival streams to the nodes are Poissonian, there is no guarantee that the flows inside the network were also Poissonian. In general, they are not.
  - except when there are no loops, i.e. a customer never re-enters a previously visited queue; then the Poisson property follows from Burke's theorem.

Stream λ_i is composed of the direct stream from the source and the split output streams from other nodes:

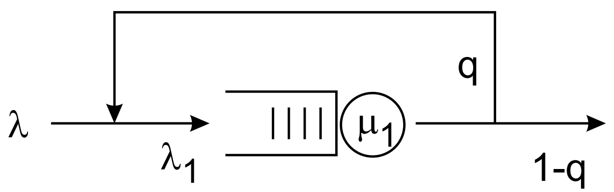
$$\boxed{\lambda_i = \lambda q_{s,i} + \sum_{j=1}^{M} \lambda_j q_{j,i}} \qquad i = 1, \ldots, M$$

The conservation laws constitute a set of linear equations, from which the λ_i can be solved.

A similar equation holds for the destination d. Since the total stream exiting the network must equal the stream entering the network, we have ($q_{s,d} = 0$):

$$\boxed{\lambda = \sum_{j=1}^{M} \lambda_j q_{j,d}}$$

Example



$$\lambda_1 = \lambda + q\lambda_1$$

$$\Rightarrow \lambda_1 = \frac{\lambda}{1-q}$$

Note. λ_1 is not Poissonian although λ is.

Jackson's theorem

- The number of customers N_i in different nodes, $i = 1, \ldots, M$, are independent.
- Queue i behaves as if the arrival stream λ_i were Poissonian.

State vector

The network state is determined by the vector $\mathbf{N} = (N_1, \ldots, N_M)$.
Its possible values are denoted by $\mathbf{n} = (n_1, \ldots, n_M)$.
The network is in state $\mathbf{n}$ when $\mathbf{N} = \mathbf{n}$, i.e. $N_1 = n_1, \ldots, N_M = n_M$.

State probability

$p(\mathbf{n}) = \mathrm{P}\{\mathbf{N} = \mathbf{n}\}$

Define $p(\mathbf{n}) = 0$, if some $n_i < 0$

Jackson's theorem

$$\boxed{p(\mathbf{n}) = p_1(n_1) \cdots p_M(n_M) = \prod_{i=1}^{M} p_i(n_i)} \qquad \text{where} \qquad \boxed{p_i(n_i) = (1 - \rho_i)\rho_i^{n_i}} \qquad \rho_i = \lambda_i / \mu_i$$

- The network behaves as if it were composed of idependent $M/M/1$ queues.
- The state probability is of the product form $\Leftrightarrow$ independence.
- If there are many customers in one of the nodes, this does not imply anything about the number of customers in other nodes.

If the service rate is state dependent $\mu_i(n_i)$, then

$$p_i(n_i) = p_i(0) \frac{\lambda_i^{n_i}}{\prod_{j=1}^{n_i} \mu_i(j)}$$

where $p_i(0)$ is determined by the normalization condition.

Proof of Jackson's theorem

Denote $\mathbf{e}_i = (0, \ldots, 0, \underbrace{1}_{\text{component } i}, 0, \ldots, 0)$.

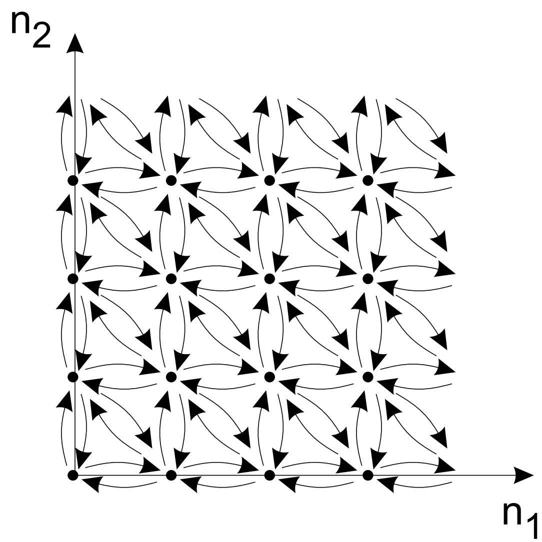
Then, for instance

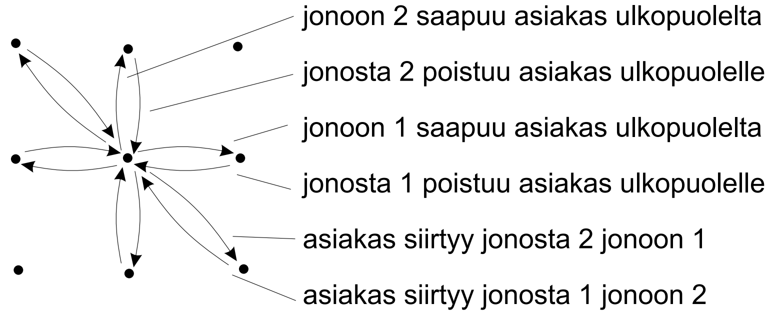
$p(\mathbf{n} - \mathbf{e}_i) = p(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_M)$

Write the globabl balance condition for state $\mathbf{n}$ (one equation for each possible state $\mathbf{n}$, $n_i \geq 0\ \forall i$):

$$\begin{aligned} \lambda\, p(\mathbf{n}) + \sum_{i=1}^{M} \mu_i 1_{n_i>0}\, p(\mathbf{n}) \;&=\; \lambda \sum_{i=1}^{M} q_{s,i}\, p(\mathbf{n} - \mathbf{e}_i) \\ &+ \sum_{i=1}^{M} q_{i,d}\, \mu_i\, p(\mathbf{n} + \mathbf{e}_i) \\ &+ \sum_{i=1}^{M} \sum_{j=1}^{M} q_{j,i}\, \mu_j\, p(\mathbf{n} + \mathbf{e}_j - \mathbf{e}_i) \end{aligned}$$

where the lhs represents the probability flow out of state $\mathbf{n}$ (in state $\mathbf{n}$, any arrival or any departure causes a transition to another state) ant the rhs the flow to state $\mathbf{n}$ (cf. the transition diagram).





Note. One could again allow state dependent service rates $\mu_i = \mu_i(n_i)$.

Proof of Jackson's theorem (continued)

Rewrite the factor $\lambda q_{s,i}$ in the first term on the rhs by means of the flow conservation equation: $\lambda q_{s,i} = \lambda_i - \sum_{j=1}^{M} \lambda_j q_{j,i}$. The equation becomes

$$\begin{aligned}
\lambda\, p(\mathbf{n}) + \sum_{i=1}^{M} \mu_i 1_{n_i>0}\, p(\mathbf{n}) \;=\; & \sum_{i=1}^{M} \lambda_i\, p(\mathbf{n}-\mathbf{e}_i) - \sum_{i=1}^{M}\sum_{j=1}^{M} \lambda_j\, q_{j,i}\, p(\mathbf{n}-\mathbf{e}_i) \\
& + \sum_{i=1}^{M} q_{i,d}\, \mu_i\, p(\mathbf{n}+\mathbf{e}_i) \\
& + \sum_{i=1}^{M}\sum_{j=1}^{M} q_{j,i}\, \mu_j\, p(\mathbf{n}+\mathbf{e}_j-\mathbf{e}_i)
\end{aligned}$$

We insert the product form solution of Jackson's theorem as a trial and show that the equation indeed is satisfied.

The following relations hold for the product form trial

$$\begin{cases}
\lambda_i\, p(\mathbf{n}-\mathbf{e}_i) = \mu_i\, 1_{n_i>0}\, p(\mathbf{n}) \\
\lambda_j\, p(\mathbf{n}-\mathbf{e}_i) = \mu_j\, p(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j) \\
\lambda_i\, p(\mathbf{n}) = \mu_i\, p(\mathbf{n}+\mathbf{e}_i)
\end{cases}$$

Substitute these relations, in this order, into the first three terms on the rhs.

Proof of Jackson's theorem (continued)

The equation takes now the form

$$\begin{aligned}\lambda\, p(\mathbf{n}) + \sum_{i=1}^{M} \mu_i 1_{n_i>0}\, p(\mathbf{n}) \;=\;& \sum_{i=1}^{M} \mu_i\, 1_{n_i>0}\, p(\mathbf{n}) - \sum_{i=1}^{M}\sum_{j=1}^{M} q_{j,i}\, \mu_j\, p(\mathbf{n}+\mathbf{e}_j-\mathbf{e}_i) \\ &+ \sum_{i=1}^{M} q_{i,d}\, \lambda_i\, p(\mathbf{n}) \\ &+ \sum_{i=1}^{M}\sum_{j=1}^{M} q_{j,i}\, \mu_j\, p(\mathbf{n}+\mathbf{e}_j-\mathbf{e}_i)\end{aligned}$$

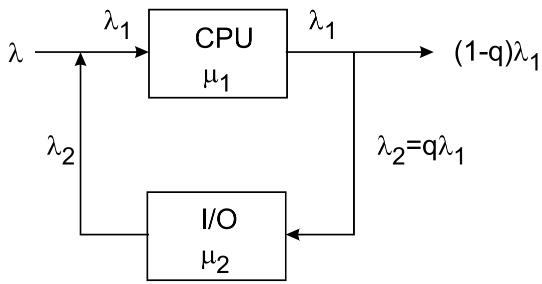
The second term on the lhs and the first term on the rhs cancel; so do the second and fourth term on the rhs. What remains is

$$\lambda\, p(\mathbf{n}) \;=\; \sum_{i=1}^{M} q_{i,d}\, \lambda_i\, p(\mathbf{n}) = \; p(\mathbf{n}) \sum_{i=1}^{M} q_{i,d}\, \lambda_i$$

which is satisfied, because the streams into and out from the network are equal, $\lambda = \sum_{i=1}^{M} \lambda_i q_{i,d}$.
Thus we have shown that the product form solution stated in Jackson's theorem indeed satisfy the global balance equations of the Markov process describing the network.

Example

$$\begin{cases} \lambda_1 = \lambda + \lambda_2 \\ \lambda_2 = q \cdot \lambda_1 \end{cases} \Rightarrow \begin{cases} \lambda_1 = \dfrac{\lambda}{1-q} \\ \lambda_2 = \dfrac{\lambda \cdot q}{1-q} \end{cases}$$



$$\rho_1 = \lambda_1/\mu_1, \qquad \rho_2 = \lambda_2/\mu_2$$

$$p(n_1, n_2) = (1-\rho_1)\rho_1^{n_1}(1-\rho_2)\rho_2^{n_2}$$

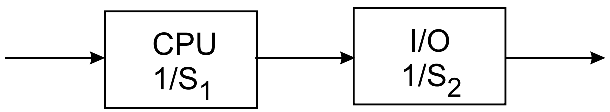
Mean queue lengths

$$\bar{N}_1 = \frac{\rho_1}{1-\rho_1}, \qquad \bar{N}_2 = \frac{\rho_2}{1-\rho_2}, \qquad \bar{N} = \bar{N}_1 + \bar{N}_1 = \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2}$$

mean time in the system

$$\bar{T} = \frac{\bar{N}}{\lambda} = \frac{\rho_1}{\lambda(1-\rho_1)} + \frac{\rho_2}{\lambda(1-\rho_2)} = \frac{\lambda_1/\mu_1}{\lambda(1-\lambda_1/\mu_1)} + \frac{\lambda_2/\mu_2}{\lambda(1-\lambda_2/\mu_2)} = \frac{S_1}{1-\lambda S_1} + \frac{S_2}{1-\lambda S_2}$$

Equivalent system



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$$\begin{cases} S_1 = \dfrac{\lambda_1}{\mu_1 \lambda} = \dfrac{1}{1-q} \cdot \dfrac{1}{\mu_1} & \text{averge CPU time} \\ S_2 = \dfrac{\lambda_2}{\mu_2 \lambda} = \dfrac{q}{1-q} \cdot \dfrac{1}{\mu_2} & \text{average I/O time} \end{cases}$$

Mean results of Jackson's networks

Assume state independent service rates μ_i.

Mean number of customers in node i

$$\bar{N}_i = \frac{\rho_i}{1-\rho_i}$$

Mean sojourn time in node i

$$\bar{T}_i = \frac{\bar{N}_i}{\lambda_i} = \frac{1}{1-\rho_i}\frac{1}{\mu_i} = \frac{1}{\mu_i - \lambda_i}$$

Mean waiting time in node i

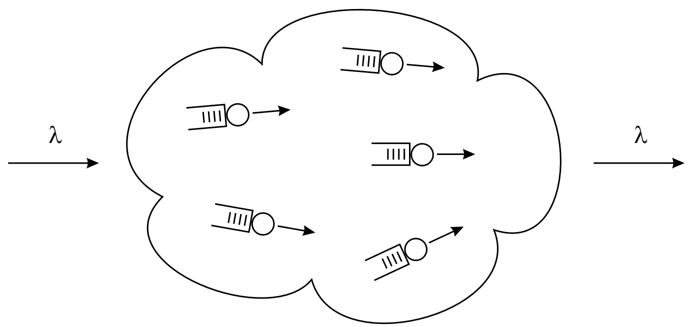
$$\bar{W}_i = \bar{T}_i - \frac{1}{\mu_i} = \frac{\rho_i}{1-\rho_i}\frac{1}{\mu_i}$$

Mean time in the network of a customer entering node i

$$\boxed{\bar{T}_{i,d} = \bar{T}_i + \sum_{j=1}^{M} q_{i,j}\bar{T}_{j,d}}$$

cf. flow conservation equations
From this set of eqs. $(i = 1, \ldots, M)$ the $\bar{T}_{i,d}$ can be solved.

Mean time in the network of a customer (average over the whole customer population)



By Little's result

$$\boxed{\bar{T} = \frac{\bar{N}}{\lambda} = \frac{1}{\lambda}\sum_{i=1}^{M}\frac{\lambda_i}{\mu_i - \lambda_i}}$$

Optimal capacity allocation

We wish to minimize the mean time $\bar{T}$ spent by customers in the network, or, equivalently, mean number of customers $\bar{N}$ in the network.

Assume that the capacities μ_i can be freely chosen except for the constraint (cost constraint) $\Sigma_{i=1}^{M} \mu_i = C$.

$$\bar{N} = \sum_{i=1}^{M} \frac{\lambda_i}{\mu_i - \lambda_i} = \min!, \qquad \sum_{i=1}^{M} \mu_i = C$$

By the method of Lagrange multipliers one minimizes

$$H = \sum_{i=1}^{M} \frac{\lambda_i}{\mu_i - \lambda_i} + x\left(\sum_{i=1}^{M} \mu_i - C\right)$$

with respect to the parameters μ_i and then determines x such that the minimum satisfies the constraint

$$\frac{\partial H}{\partial \mu_i} = -\frac{\lambda_i}{(\mu_i - \lambda_i)^2} + x = 0 \quad \Rightarrow \quad \mu_i = \lambda_i + (\lambda_i / x)^{1/2}$$

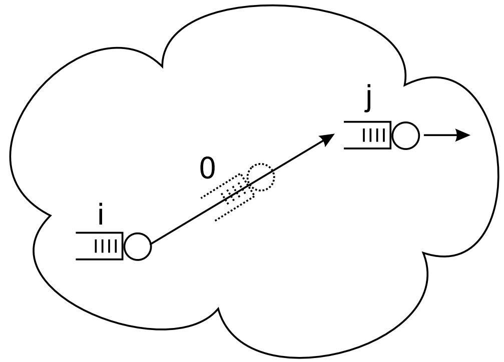
By inserting this into the constraint condition, one gets

$$\frac{1}{\sqrt{x}} = \frac{C - \sum_j \lambda_j}{\sum_j \sqrt{\lambda_j}} \quad \Rightarrow \quad \boxed{\mu_i = \lambda_i + \frac{\sqrt{\lambda_i}}{\sum_j \sqrt{\lambda_j}}\left(C - \sum_j \lambda_j\right)}$$

One first allocates the mandatory capacity λ_i; the excess money is distributed in relation to the $\sqrt{\lambda_i}$.

Arrival theorem of open networks (Random Observer Property, cf. PASTA)

In on open network a customer entering any queue sees the same state probabilities (the probability the system is in a state just before the arrival) are the same as the equilibrium probabilities $p(\mathbf{n})$.



Proof. Consider a customer transiting from queue i to queue j. Insert between these queues a virtual queue 0 with a very high service rate μ_0.

In the limit $\mu_0 \to \infty$, the added queue does not affect the system at all: the customers transiting from queue i to queue j spend an infinitesimal time in the added virtual queue.

The virtual queue, however, enables "catching" the transiting customer. The transition occurs precisely in the short interval when there is customer in queue 0, i.e. when $N_0(t) = 1$. The state distribution seen by the transiting customer is the distribution of the other queues (than queue 0) conditioned on $N_0 = 1$.

Now make use of the fact that also the extended system is a Jackson network with a product form solution. Denote the state vector of the extended system by $\mathbf{n}'$, i.e. $\mathbf{n}' = (n_0, n_1, \ldots, n_M)$. It holds $p'(\mathbf{n}') = \mathrm{P}\{\mathbf{N}' = \mathbf{n}'\} = p_0(n_0)p_1(n_1)\cdots p_M(n_M) = p_0(n_0)p(\mathbf{n})$.

$$\mathrm{P}\{N_1 = n_1, \ldots, N_M = n_M \,|\, N_0 = 1\} = \frac{\mathrm{P}\{N_0 = 1, N_1 = n_1, \ldots, N_M = n_M\}}{\mathrm{P}\{N_0 = 1\}} = p(\mathbf{n})$$

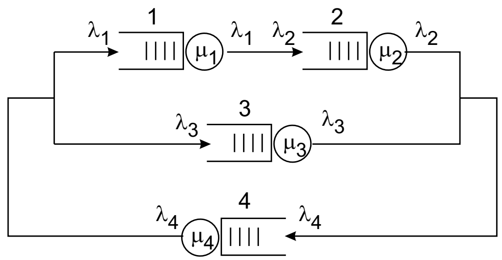
Closed queueing networks (Gordon and Newell networks)

A closed queueing network consists of M nodes. In contrast to an open network, there is no external source or sink. There is a constant population of K customers in the network.

- Each node i is a FIFO queue, where the service time is drawn independently from the distribution Exp(μ_i). Again we could have state dependent service rates $\mu_i(n_i)$.
- A customer departing from queue i chooses queue j next with probability $q_{i,j}$.

The customer streams through different nodes satisfy the conservation law

$$\boxed{\lambda_i = \sum_{j=1}^{M} \lambda_j q_{j,i}} \qquad i = 1, \ldots, M$$



These constitute a homogeneous linear set of equations. One equation is linearly dependent on the others, and the solution is determined uniquely up to a constant factor.

Let $(\hat{\lambda}_1, \ldots, \hat{\lambda}_M)$ be a solution. The general solution is of the form $\alpha \cdot (\hat{\lambda}_1, \ldots, \hat{\lambda}_M)$, where α is a constant. Which value of α corresponds to the actual streams remains so far undetermined. This will be fixed later (see MVA). Let this value be denoted by $\hat{\alpha}$, i.e. $\lambda_i = \hat{\alpha}\hat{\lambda}_i$.

Denote $\hat{\rho}_i = \hat{\lambda}_i/\mu_i$. These quantities are correspondingly proportional to the real loads of the queues $\rho_i = \lambda_i/\mu_i$, viz. $\rho_i = \hat{\alpha}\hat{\rho}_i$.

The theorem of Gordon and Newell

The equilibrium probabilities of a closed queueing network are

$$p(\mathbf{n}) = \begin{cases} \dfrac{1}{G(K,M)} \prod_{i=1}^{M} \hat{\rho}_i^{n_i}, & \text{when } \Sigma_i n_i = K \\ 0, & \text{when } \Sigma_i n_i \neq K \end{cases} \quad \text{where} \quad G(K,M) = \sum_{\mathbf{n}:\, \sum_i n_i = K} \prod_{i=1}^{M} \hat{\rho}_i^{n_i}$$

The proof is similar to that in open networks. The details will be omitted.

The probability distribution is again of product form in the allowed region $\Sigma_i n_i = K$ (but not everywhere!).

Note. Although the factors $\hat{\rho}_i$ contain an undetermined coefficient, the solution itself is unique, as the same factor in power K appears in the product and the norm factor in the denominator.

Arrival theorem in closed networks (Lavenberg)

In a closed network with K customers, the state probabilities seen by a customer entering any node are the same as the equilibrium probabilities $p[K-1](\mathbf{n})$ in a network with $K-1$ customers (Compare with the state distribution in the Engset system; an arbitrary customer is as if he were an "external observer").

The theorem can be proven in the same way by means of a virtual queue as in the case of the open network. Details will be omitted.

Mean value analysis, MVA (Reiser and Lavenberg)

Our objective now is to find the mean number of customers $\bar{N}_i[K]$ and sojourn times $\bar{T}_i[K]$ as well as the absolute values of customer streams λ_i through different queues.

The analysis is centrally based on the arrival theorem. The calculation proceeds recursively, incrementing the customer population in the network step by step. Therefore, the total customer population is explicitly indicated in brackets.

The mean sojourn time in queue i is

$$\bar{T}_i[K] = \underbrace{\frac{1}{\mu_i}}_{\text{own service time}} + \underbrace{\bar{N}_i^*[K] \cdot \frac{1}{\mu_i}}_{\text{the time it takes to serve the customers ahead}}$$

where $\bar{N}_i^*[K]$ is the mean occupancy seen by a customer arriving at queue i.
By the arrival theorem we have

$$\bar{N}_i^*[K] = \bar{N}_i[K-1]$$

where $\bar{N}_i[K-1]$ is the mean occupancy calculated from the equilibrium distribution. Thus

$$\boxed{\bar{T}_i[K] = (1 + \bar{N}_i[K-1]) \cdot \frac{1}{\mu_i}}$$

Mean value analysis (continued)

The mean occupancy in queue i is

$$\boxed{\bar{N}_i[K] = K \cdot \frac{\hat{\lambda}_i \cdot \bar{T}_i[K]}{\sum_{j=1}^{M} \hat{\lambda}_j \cdot \bar{T}_j[K]}}$$

Proof. The real customer streams are $\lambda_i = \hat{\alpha}\hat{\lambda}_i$. By expanding the above expression by $\hat{\alpha}$ and by applying Little's result we see that

$$K \cdot \frac{\hat{\lambda}_i \cdot \bar{T}_i[K]}{\sum_{j=1}^{M} \hat{\lambda}_j \cdot \bar{T}_j[K]} = K \cdot \frac{\lambda_i[K] \cdot \bar{T}_i[K]}{\sum_{j=1}^{M} \lambda_j[K] \cdot \bar{T}_j[K]} = K \cdot \frac{\bar{N}_i[K]}{\sum_{j=1}^{M} \bar{N}_j[K]} = K \cdot \frac{\bar{N}_i[K]}{K} = \bar{N}_i[K]$$

Using Little's result in the reverse direction we get the real customer stream through queue i:

$$\boxed{\lambda_i[K] = \frac{\bar{N}_i[K]}{\bar{T}_i[K]} = K \cdot \frac{\hat{\lambda}_i}{\sum_{j=1}^{M} \hat{\lambda}_j \cdot \bar{T}_j[K]}}$$

MVA algorithm

The results can be collected as the following recursive method.

Start of the recursion:

$$\boxed{\bar{N}_i[0] = 0}$$

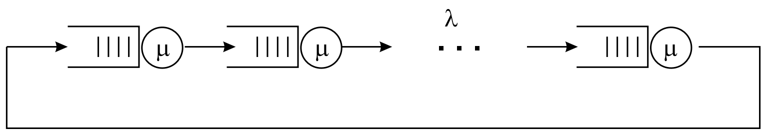
In an empty network all mean queue lengths are zero.

Recursion step:

$$\begin{aligned}
\bar{T}_i[K] &= (1 + \bar{N}_i[K-1]) \cdot \frac{1}{\mu_i} \\
\bar{N}_i[K] &= K \cdot \frac{\hat{\lambda}_i \cdot \bar{T}_i[K]}{\sum_{j=1}^{M} \hat{\lambda}_j \cdot \bar{T}_j[K]} \\
\lambda_i[K] &= \frac{\bar{N}_i[K]}{\bar{T}_i[K]}
\end{aligned}$$

In the equation in the middle, the $\hat{\lambda}_i$ are any solution to the flow equations $\lambda_i = \Sigma_j \lambda_j q_{j,i}$.

Example 1. Cyclic network



$$\mu_1 = \mu_2 = \cdots = \mu_M = \mu$$

A solution to the flow equations is:
$\hat{\lambda}_1 = \hat{\lambda}_2 = \cdots = \hat{\lambda}_M = 1$

Since all the queues are identical we can drop the node index. Now the recursion equations read

$$\begin{cases} \bar{T}[K] &= (1 + \bar{N}[K-1]) \cdot \frac{1}{\mu} \\ \bar{N}[K] &= K/M \\ \lambda[K] &= \bar{N}[K]/\bar{T}[K] \end{cases}$$

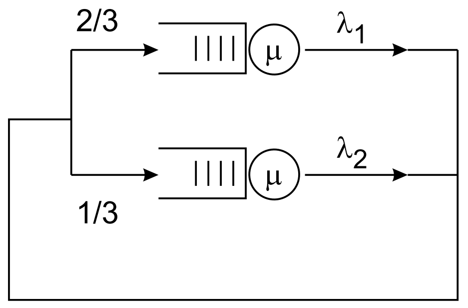
Starting from the initial value $\bar{N}[0] = 0$, one solves the mean values for progressively greater populations

$$\begin{cases} \bar{T}[1] &= \frac{1}{\mu} \\ \bar{N}[1] &= \frac{1}{M} \\ \lambda[1] &= \frac{1}{M}\,\mu \end{cases} \quad \begin{cases} \bar{T}[2] &= \frac{M+1}{M}\frac{1}{\mu} \\ \bar{N}[2] &= \frac{2}{M} \\ \lambda[2] &= \frac{2}{M+1}\,\mu \end{cases} \quad \begin{cases} \bar{T}[3] &= \frac{M+2}{M}\frac{1}{\mu} \\ \bar{N}[2] &= \frac{3}{M} \\ \lambda[2] &= \frac{3}{M+2}\,\mu \end{cases} \quad \ldots \quad \begin{cases} \bar{T}[K] &= \frac{M+K-1}{M}\frac{1}{\mu} \\ \bar{N}[K] &= \frac{K}{M} \\ \lambda[K] &= \frac{K}{M+K-1}\,\mu \end{cases}$$

When $K \ll M$ then $\lambda[K] \approx \frac{K}{M}\mu$ (mean time of a full cycle is M/μ, there are K customers).

When $K \gg M$ then $\lambda[K] \approx \mu$ (all queues full; customers depart on av. at intervals $1/\mu$).

Example 2.



$K = 3$

A solution to the flow equations is:

$\hat{\lambda}_1 = 2,\ \hat{\lambda}_2 = 1$

Starting from the initial values $\bar{N}_1[0] = \bar{N}_2[0] = 0$, one solves the mean values for progressively greater populations

$$K=1 \quad \begin{cases} \bar{T}_1[1] = \frac{1}{\mu} \\ \bar{N}_1[1] = 1 \cdot \frac{2/\mu}{2/\mu+1/\mu} = \frac{2}{3} \\ \lambda_1[1] = \frac{2}{3}\,\mu \end{cases} \qquad \begin{cases} \bar{T}_2[1] = \frac{1}{\mu} \\ \bar{N}_2[1] = 1 \cdot \frac{1/\mu}{2/\mu+1/\mu} = \frac{1}{3} \\ \lambda_2[1] = \frac{1}{3}\,\mu \end{cases}$$

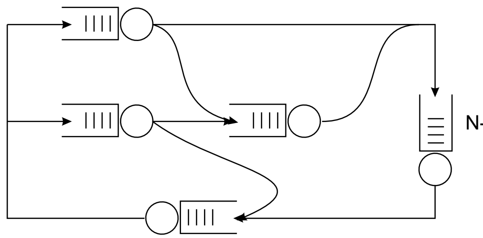
$$K=2 \quad \begin{cases} \bar{T}_1[2] = (1+\frac{2}{3})\,\frac{1}{\mu} = \frac{5}{3}\,\frac{1}{\mu} \\ \bar{N}_1[2] = 2 \cdot \frac{2\cdot 5/3}{2\cdot 5/3+1\cdot 4/3} = \frac{10}{7} \\ \lambda_1[2] = \frac{6}{7}\,\mu \end{cases} \qquad \begin{cases} \bar{T}_2[2] = (1+\frac{1}{3})\,\frac{1}{\mu} = \frac{4}{3}\,\frac{1}{\mu} \\ \bar{N}_2[2] = 2 \cdot \frac{1\cdot 4/3}{2\cdot 5/3+1\cdot 4/3} = \frac{4}{7} \\ \lambda_2[2] = \frac{3}{7}\,\mu \end{cases}$$

$$K=3 \quad \begin{cases} \bar{T}_1[3] = (1+\frac{10}{7})\,\frac{1}{\mu} = \frac{17}{7}\,\frac{1}{\mu} \\ \bar{N}_1[3] = 3 \cdot \frac{2\cdot 17/7}{2\cdot 17/7+1\cdot 11/7} = \frac{34}{15} \\ \lambda_1[3] = \frac{14}{15}\,\mu \end{cases} \qquad \begin{cases} \bar{T}_2[3] = (1+\frac{4}{7})\,\frac{1}{\mu} = \frac{11}{7}\,\frac{1}{\mu} \\ \bar{N}_2[3] = 3 \cdot \frac{1\cdot 11/7}{2\cdot 17/7+1\cdot 11/7} = \frac{11}{15} \\ \lambda_2[3] = \frac{7}{15}\,\mu \end{cases}$$

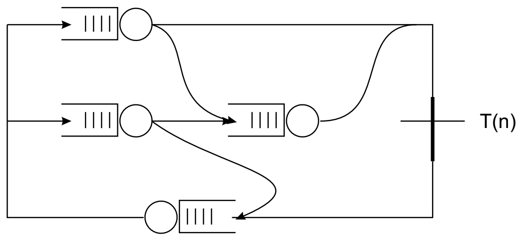
Norton's theorem

For queueing networks, one can derive the same kind of Norton's theorem that is used in the analysis of linear circuits. In the case of queueing networks, the theorem can be proven by using the known equilibrium distribution.

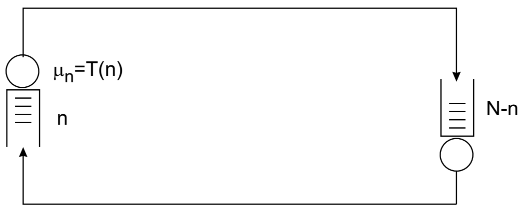
When we are interested in the behaviour of just a part of the network, say queue i, the rest of the network can be replaced by an "equivalent queue".



N customers in the network.
$N - n$ customers in queue i.
n customers in other part of the network.



Calculate the throughput (stream) $T(n)$ for the "short circuited" system as a function of the number of customers n.



The equivalent queue replacing the other part of the network has a state dependent service rate $T(n)$. Queue i behaves exactly as in the original network.