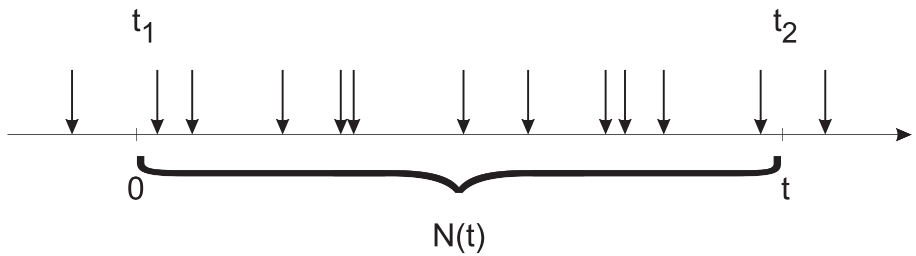
Poisson process

General

Poisson process is one of the most important models used in queueing theory.

- Often the arrival process of customers can be described by a Poisson process.
- In teletraffic theory the "customers" may be calls or packets. Poisson process is a viable model when the calls or packets originate from a large population of independent users.

In the following it is instructive to think that the Poisson process we consider represents discrete arrivals (of e.g. calls or packets).



Mathematically the process is described by the so called counter process N_t or $N(t)$. The counter tells the number of arrivals that have occurred in the interval $(0, t)$ or, more generally, in the interval (t_1, t_2).

$$\begin{cases} N(t) = \text{number of arrivals in the interval } (0,t) & \text{(the stochastic process we consider)} \\ N(t_1,t_2) = \text{number of arrival in the interval } (t_1,t_2) & \text{(the increment process } N(t_2)-N(t_1)\text{)} \end{cases}$$

General (continued)

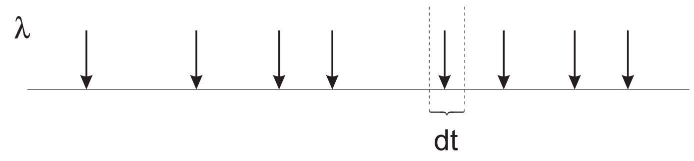
A Poisson process can be characterized in different ways:

- Process of independent increments
- Pure birth process
  - the arrival intensity λ (mean arrival rate; probability of arrival per time unit
- The "most random" process with a given intensity λ

Definition

The Poisson process can be defined in three different (but equivalent) ways:

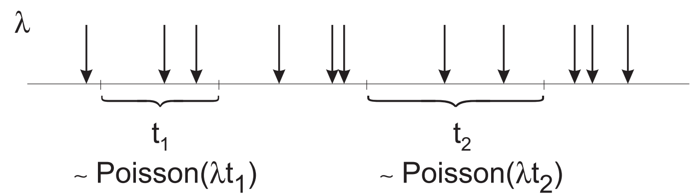
1. Poisson process is a pure birth process:
   In an infinitesimal time interval dt there may occur only one arrival. This happens with the probability λdt independent of arrivals outside the interval.



2. The number of arrivals $N(t)$ in a finite interval of length t obeys the Poisson(λt) distribution,

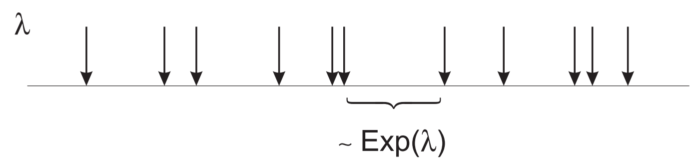
$$\mathrm{P}\{N(t) = n\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

   Moreover, the number of arrivals $N(t_1, t_2)$ and $N(t_3, t_4)$ in non-overlapping intervals ($t_1 \le t_2 \le t_3 \le t_4$) are independent.



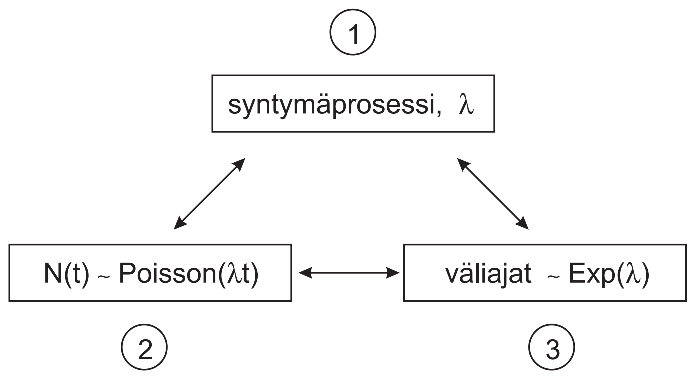
3. The interarrival times are independent and obey the Exp(λ) distribution:

$$\mathrm{P}\{\text{interarrival time} > t\} = e^{-\lambda t}$$



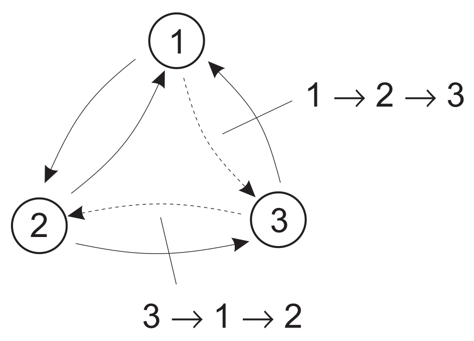
The equivalence of the definitions

The three definitions are equivalent:



In the following we show the equivalence by showing the implications in the direction of the solid arrows. Then any of the three properties implies the other two ones.

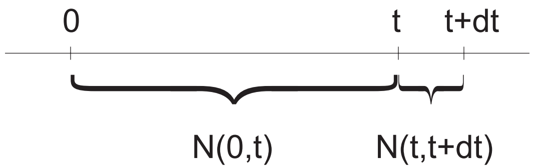
In fact, the implication $2 \rightarrow 1$ is not necessary for proving the equivalence (as it follows from the implications $2 \rightarrow 3$ and $3 \rightarrow 1$), but it can be shown very easily directly.



Proof of the equivalence: part $(1 \rightarrow 2)$

We wish to show that property 1 implies property 2 (essentially this was already shown when discussing the pure birth process).

Assume that arrivals in different intervals are independent and



$$\mathrm{P}\{\text{arrival in } (t, t+dt)\} = \lambda \cdot dt$$

Consider the generating function of the counter $\mathcal{G}_t(z)$:

$$\mathcal{G}_t(z) = \mathrm{E}[z^{N(0,t)}]$$

$$\begin{aligned}\mathcal{G}_{t+dt}(z) &= \mathrm{E}[z^{N(0,t+dt)}] = \mathrm{E}[z^{N(0,t)+N(t,t+dt)}] = \underbrace{\mathrm{E}[z^{N(0,t)}]}_{\mathcal{G}_t(z)}\ \underbrace{\mathrm{E}[z^{N(t,t+dt)}]}_{(1-\lambda\cdot dt)z^0+\lambda\cdot dt\cdot z^1} \\ &= \mathcal{G}_t(z) - \lambda dt(1-z)\mathcal{G}_t(z)\end{aligned}$$

$$\frac{\mathcal{G}_{t+dt}(z)-\mathcal{G}_t(z)}{dt} = \lambda(z-1)\mathcal{G}_t(z) \quad \Rightarrow \quad \frac{d}{dt}\mathcal{G}_t(z) = \lambda(z-1)\mathcal{G}_t(z)$$

$$\frac{d}{dt}\log\mathcal{G}_t(z) = \lambda(z-1) \quad \Rightarrow \quad \log\mathcal{G}_t(z) - \underbrace{\log\mathcal{G}_0(z)}_{0} = \lambda(z-1)t$$

$$\boxed{\mathcal{G}_t(z) = e^{(z-1)\lambda t}}$$ generating function of the Poisson distribution

Proof of the equivalence: part $(2 \rightarrow 1)$

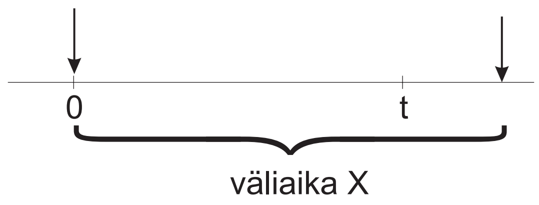
Assume that $\mathrm{P}\{N(t) = n\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$. Then

$$\begin{cases} \mathrm{P}\{N(dt) = 0\} = e^{-\lambda dt} = 1 - \lambda \cdot dt + o(dt) \\ \mathrm{P}\{N(dt) = 1\} = \frac{\lambda \cdot dt}{1!} e^{-\lambda dt} = \lambda \cdot dt + o(dt) \end{cases}$$

Moreover, since property 2 assumes independence of arrivals in non-overlapping intervals an arrival in interval dt occurs independently of arrivals outside the interval.

Proof of the equivalence: part $(2 \rightarrow 3)$

Consider the time interval X between two arrivals:



$$\{X > t\} \quad \equiv \quad \{N(t) = 0\} \qquad \text{(the events are equivalent)}$$

$$\mathrm{P}\{X > t\} \;=\; \mathrm{P}\{N(t) = 0\} \;=\; e^{-\lambda t}$$

$$\Rightarrow \; X \sim \mathrm{Exp}(\lambda)$$

Proof of the equivalence: part $(3 \rightarrow 1)$

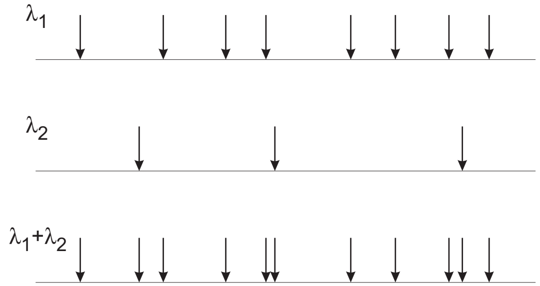
It was noted already in considering the exponential distribution: If $X \sim \mathrm{Exp}(\lambda)$ then the probability that the period ends (an arrival occurs) in the interval dt is $\lambda \cdot dt + \mathcal{O}(dt)$.

Properties of the Poisson process

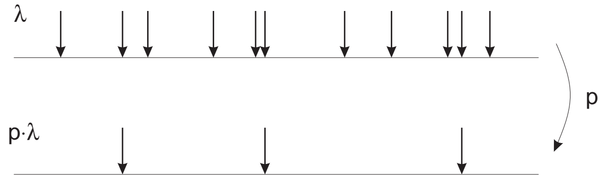
The Poisson process has several interesting (and useful) properties:

1. Conditioning on the number of arrivals. Given that in the interval $(0, t)$ the number of arrivals is $N(t) = n$, these n arrivals are independently and uniformly distributed in the interval.
   - One way to generate a Poisson process in the interval $(0, t)$ is as follows:
     - draw the total number of arrivals n from the Poisson(λt) distribution
     - for each arrival draw its position in the interval $(0, t)$ from the uniform distribution, independently of the others

2. Superposition. The superposition of two Poisson processes with intensities λ_1 and λ_2 is a Poisson process with intensity $\lambda = \lambda_1 + \lambda_2$.

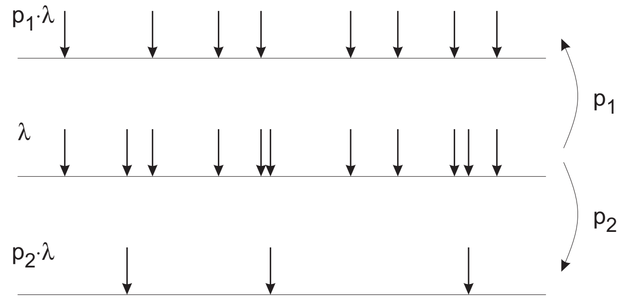


3. Random selection. If a random selection is made from a Poisson process with intensity λ such that each arrival is selected with probability p, independently of the others, the resulting process is a Poisson process with intensity $p\lambda$.



Properties of the Poisson process (continued)

4. Random split. If a Poisson process with intensity λ is randomly split into two subprocesses with probabilities p_1 and p_2, where $p_1 + p_2 = 1$, then the resulting processes are independent Poisson processes with intensities $p_1\lambda$ ja $p_2\lambda$.
(This result allows an straight forward generalization to a split into more than two subprocesses.)



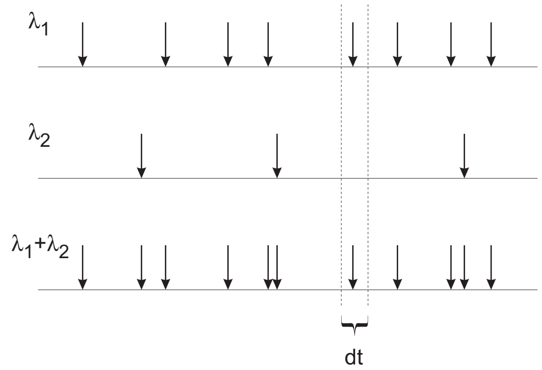
5. PASTA. The Poisson process has the so called PASTA property (Poisson Arrivals See Time Averages): for instance, customers with Poisson arrivals see the system as if they came into the system at a random instant of time (despite they induce the evolution of the system).

We prove some of these properties. The proof of property 1 is left as an exercise.

For the proofs we may use any of the given tree definitions of the Poisson process as we find most convenient.

Proof (property 2, superposition)

The probability that an arrival occurs from process 1 in the interval dt is $\lambda_1 \cdot dt$ independent of the arrivals outside the interval. Similarly, the arrival probability from process 2 is λ_2 dt.



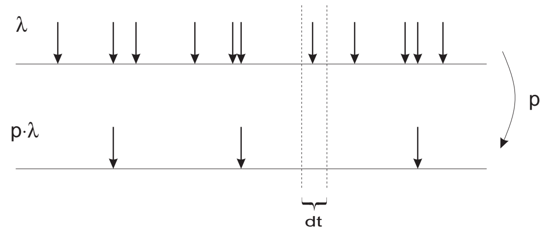
⇒ In the superposed process the probability for an arrival in the interval dt is $(\lambda_1 + \lambda_2)dt$ independent of arrivals outside the interval.

⇒ The superposition is a Poisson process with intensity $\lambda_1 + \lambda_2$.

Proof (property 3, random selection)

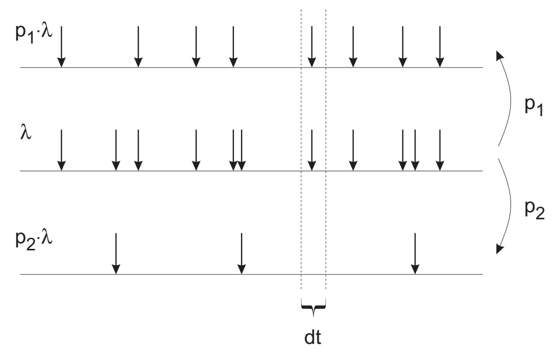
The probability that an arrival occurs from the original process in the interval dt is $\lambda \cdot dt$ independent of the arrivals outside the interval.

After the random selection the probability for an arrival in the interval dt is $p \cdot \lambda \cdot dt$ (independent of the arrivals outside the interval).



⇒ The process of the selected arrivals is a Poisson process with intensity $p \cdot \lambda$.

Proof (property 4, random split)

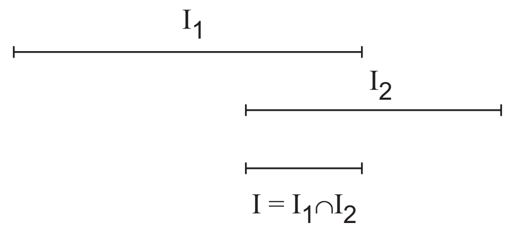


Both of the subprocesses resulting from the split represent a random selection of the original process and are thus Poisson processes with intensities $p_i\lambda$.

It remains to prove the independence of the processes. Let

$$\begin{cases} N_1(I_1) = & \text{number of arrivals from subprocess 1 in the interval } I_1 \\ N_2(I_2) = & \text{number of arrivals from subprocess 2 in the interval } I_2 \end{cases}$$

Denote $I = I_1 \cap I_2$



$$\begin{cases} N_1(I_1) = N_1(I) + N_1(I_1 \cap \bar{I}_2) \\ N_2(I_2) = N_2(I) + N_2(I_2 \cap \bar{I}_1) \end{cases}$$

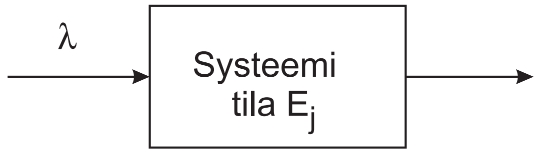
Arrivals in non-overlapping intervals $I_1 \cap \bar{I}_2$ and $I_2 \cap \bar{I}_1$ are certainly independent.

There may be dependence only between $N_1(I)$ and $N_2(I)$. But these represent the random split of the total number of arrivals from the original process, with distribution Poisson$(\lambda|I|)$, into two sets; the sizes of these sets were shown to be independent in considering the properties of the Poisson distribution.

PASTA (Poisson Arrivals See Time Averages)

The PASTA property is one of the central tools in queueing theory. Sometimes this property is referred to as ROP (Random Observer Property).

Consider an arbitrary system which spends its time in different states E_j.



Arrivals to the system constitute a Poisson process with intensity λ. These arrivals induce state transitions in the system.

In equilibrium, we may associate with each state E_j two different probabilities:

1. The probability of the state as seen by an outside random observer

   π_j = probability that the system is in the state E_j at a random instant

2. The probability of the state seen by an arriving customer

   π_j^* = probability that the system is in the state E_j just before (a randomly chosen) arrival

In general, $\boxed{\pi_j \neq \pi_j^*}$

PASTA (continued)

Example. Your own PC (one customer, one server)

$$\begin{cases} E_0 &= \text{PC free} \\ E_1 &= \text{PC occupied} \end{cases}$$

$$\begin{cases} \pi_0^* &= 1 \qquad \text{(your own PC is always free when you need it)} \\ \pi_1^* &= 0 \end{cases}$$

$$\begin{cases} \pi_0 &= \text{proportion of time the PC is free } (< 1) \\ \pi_1 &= \text{proportion of time the PC is occupied } (> 0) \end{cases}$$

Note, in this case the arrival process is not Poisson; when an arrival has occurred (i.e. you have started to work with you PC) for a while it's unlikely that another arrival occurs (i.e. you have stopped the previous session and started a new one). Thus the arrivals at different times ane not independent.

PASTA (continued)

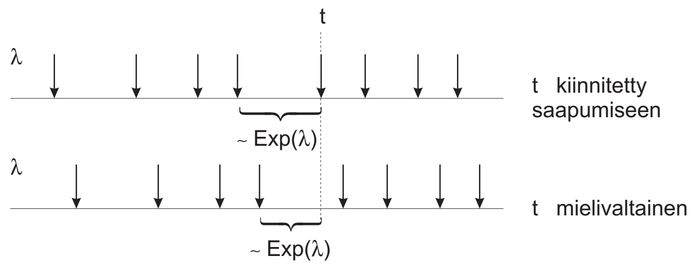
In the case of a Poisson arrival process it holds

$$\boxed{\pi_j = \pi_j^*}$$

Proof

The arrival history before the instant of consideration, irrespective whetehr we are considering a random instant or an arrival instant, are stochastically the same:
a sequence of arrivals with exponentially distributed interarrival times.

This follows from the memoryless property of the exponential distribution. The remaining time to the next arrival has the same exponential distribution irrespective of the time that has already elapsed since the previous arrival (the same holds also in reversed time, i.e. looking backwards).

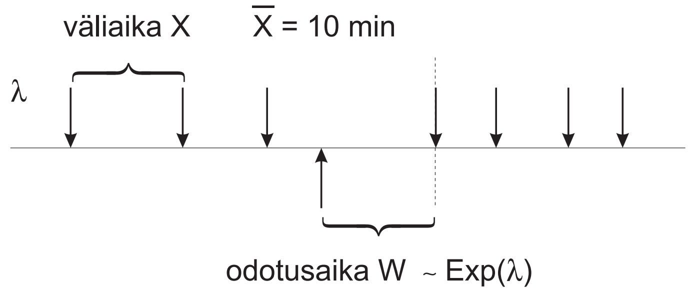


Since the stochastic characterization of the arrival process before the instant of consideration is the same, irrespective how the instant has been chosen) the state distributions of the system (induced by the past arrivals processes) at the instant of consideration must be the same in both cases.

The hitchhiker's paradox

The setting of the paradox is the following

- Cars are passing a point of a road according to a Poisson process.
- The mean interval between the cars is 10 min.
- A hitchhiker arrives to the raodside point at random instant of time.
- What is the mean waiting time $\bar{W}$ until the next car.



The interarrival times in a Poisson process are exponentially distributed. From the memoryless property of the exponential distribution it follows that the (residual) time to the next arrival has the same Exp(λ) distribution and the expected time is thus

$$\bar{W} = 10 \text{ min}$$

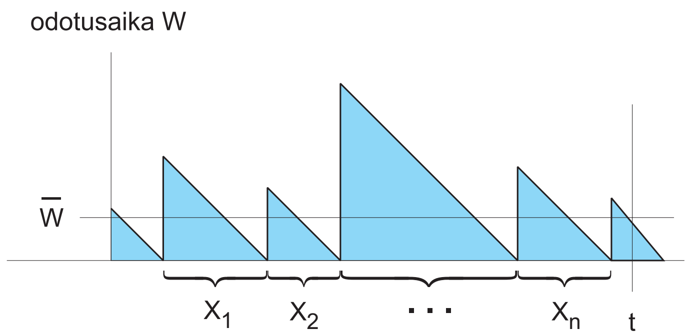
This appears paradoxical. Why isn't the expected time 5 min? Is there something wrong? Answer: No, the expected time is indeed $\bar{W} = 10$ min.

Explanation for the hitchhiker's paradox

In short, the explanation of the paradox lies therein that the hitchhiker's probability to arrive during a long interarrival interval is greater than during a short interval.

Given the interarrival interval, within that interval the arrival instant of the hitchhiker is uniformly distributed and the expected waiting time is one half of the total duration of the interval. The point is that in the selection by the random instant the long intervals are more frequently represented than the short ones (with a weight proportional to the length of the interval).

Consider a long period of time t. The waiting time to the next car arrival $W(\tau)$ as the function of the arrival instant of the hitchhiker τ is represented by the sawtooth curve in the figure. The mean waiting time is the average value of the curve.



$$\bar{W} = \frac{1}{t}\int_0^t W(\tau)d\tau \approx \frac{1}{t}\sum_{i=1}^{n}\frac{1}{2}X_i^2$$ (sum of the areas of the triangles; X_i is the interarrival time)

As $t \to \infty$ the number of the triangles n tends to $t/\bar{X}$.

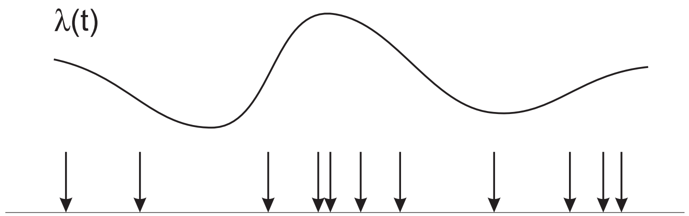
$$\boxed{\bar{W} = \frac{1}{\bar{X}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}X_i^2 = \frac{1}{2}\frac{\overline{X^2}}{\bar{X}}}$$

For exponential distribution

$$\overline{X^2} = (\bar{X})^2 + \underbrace{\mathrm{V}[X]}_{(\bar{X})^2} = 2(\bar{X})^2, \qquad \text{thus } \bar{W} = \bar{X}$$

Inhomogeneous Poisson process

Thus far we have considered a Poisson process with a constant intensity λ. This can be generalized to a so called inhomogeneous Poisson process by letting the intensity to vary in time $\lambda(t)$. (Note. $\lambda(t)$ is a deterministic function of time.)



The probability of an arrival in a short interval of time $(t, t+dt)$ is now $\lambda(t)dt + o(dt)$.

- The probability for more than one arrivals is of the order $o(dt)$
- The expected number of arrivals in the interval $(t, t+dt)$ is

$$\mathrm{E}[N(t, t+dt)] = \sum_{n=0}^{\infty} n \cdot \mathrm{P}\{n \text{ arrivals in } (t, t+dt)\} = \lambda(t)dt + o(dt)$$

- Correspondingly, the expected number of arrivals in a finite interval $(0, t)$ is

$$\mathrm{E}[N(0,t)] = \mathrm{E}[\int_0^t N(u, u+du)] = \int_0^t \mathrm{E}[N(u, u+du)] = \int_0^t \lambda(u)du$$

(The expectation of a sum is always the sum of the expectations of individual terms, therefore the order of integration and expectation can be interchanged.)

Inhomogeneous Poisson process (continued)

In the same way as in the case of an ordinary homogeneous Poisson process, we can derive a differential equation for the generating function $\mathcal{G}_t(z)$ of the counter process $N(t)$ (number of arrivals in $(0,t)$) of an inhomogeneous Poisson process,

$$\frac{d}{dt}\mathcal{G}_t(z) = (z-1)\lambda(t)\mathcal{G}_t(z) \quad \Rightarrow \quad \frac{d}{dt}\log\mathcal{G}_t(z) = (z-1)\lambda(t)$$

from which we get by integration

$$\boxed{\mathcal{G}_t(z) = e^{(z-1)\int_0^t \lambda(u)du}}$$

Denote the expected number of arrivals in $(0,t)$ by $a(t)$

$$a(t) = \mathrm{E}[N(t)] = \int_0^t \lambda(u)du$$

We see that $\mathcal{G}_t(z)$ is the generating function of a random variable with Poisson distribution. Thus,

$$\boxed{N(t) \sim \mathrm{Poisson}(a(t))}$$

Properties of an inhomogeneous Poisson process

Analogously with homogeneous Poisson process, the inhomogeneous Poisson process can be shown to have the following properties:

1. Conditioning on the number. Given the total number of arrivals $N(t) = n$ in the interval $(0, t)$ from an inhomogeneous Poisson process, the arrival instants of these n arrivals are distributed independently in the interval $(0, t)$ with the density function $\lambda(t)/\int_0^t \lambda(u)du$.

2. Superposition. The superposition of two inhomogeneous Poisson processes with intensities $\lambda_1(t)$ and $\lambda_2(t)$ is an inhomogeneous Poisson process with intensity $\lambda(t) = \lambda_1(t) + \lambda_2(t)$.

3. Random selection. A random selection from an inhomogeneous Poisson process with intensity $\lambda(t)$ such that each arrival is selected, independent of the others, with the probability $p(t)$ (note, may depend on time) results in an inhomogeneous Poisson process with intensity $p(t)\lambda(t)$.

4. Random split. If an inhomogeneous Poisson process with intensity $\lambda(t)$ is randomly split into two subprocesses with the probabilities $p_1(t)$ and $p_2(t)$, where $p_1(t) + p_2(t) = 1$, then the resulting subprocesses are independent inhomogeneous Poisson processes with intensities $p_1(t)\lambda(t)$ and $p_2(t)\lambda(t)$.

Properties of an inhomogeneous Poisson process (continued)

4. Random split (continued)