Birth-death processes

General

A birth-death (BD process) process refers to a Markov process with

- a discrete state space
- the states of which can be enumerated with index $i=0,1,2,...$ such that
- state transitions can occur only between neighbouring states, $i \rightarrow i + 1$ or $i \rightarrow i - 1$

Transition rates

$$q_{i,j} = \begin{cases} 
\lambda_i & \text{when } j = i + 1 \\
\mu_i & \text{if } j = i - 1 \\
0 & \text{otherwise}
\end{cases}$$

probability of death in interval $\Delta t$ on $\lambda_i \Delta t$

probability of birth in interval $\Delta t$ on $\mu_i \Delta t$

when the system is in state $i$
The equilibrium probabilities of a BD process

We use the method of a cut = global balance condition applied on the set of states 0, 1, \ldots, k.

In equilibrium the probability flows across the cut are balanced (net flow = 0)

\[ \lambda_k \pi_k = \mu_{k+1} \pi_{k+1} \quad k = 0, 1, 2, \ldots \]

We obtain the recursion

\[ \pi_{k+1} = \frac{\lambda_k}{\mu_{k+1}} \pi_k \]

By means of the recursion, all the state probabilities can be expressed in terms of that of the state 0, \( \pi_0 \),

\[ \pi_k = \frac{\lambda_{k-1} \lambda_{k-1} \cdots \lambda_0}{\mu_k \mu_{k-1} \cdots \mu_1} \pi_0 = \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}} \pi_0 \]

The probability \( \pi_0 \) is determined by the normalization condition \( \pi_0 \)

\[ \pi_0 = \frac{1}{1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \cdots} = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}} \]
The time-dependent solution of a BD process

Above we considered the equilibrium distribution $\pi$ of a BD process.

Sometimes the state probabilities at time 0, $\pi(0)$, are known
- usually one knows that the system at time 0 is precisely in a given state $k$; then $\pi_k(0) = 1$ and $\pi_j(0) = 0$ when $j \neq k$
and one wishes to determine how the state probabilities evolve as a function of time $\pi(t)$
- in the limit we have $\lim_{t \to \infty} \pi(t) = \pi$.

This is determined by the equation

$$\frac{d}{dt} \pi(t) = \pi(t) \cdot Q$$

where

$$Q = \begin{pmatrix}
-\lambda_0 & \lambda_0 & 0 & \ldots & \ldots \\
\mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \ldots \\
0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 \\
\vdots & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 \\
\vdots & \vdots & 0 & \mu_4 & -(\lambda_4 + \mu_4)
\end{pmatrix}$$
The time-dependent solution of a BD process (continued)

\[
\begin{align*}
\frac{d\pi_i(t)}{dt} &= -\left( \lambda_i + \mu_i \right) \pi_i(t) + \lambda_{i-1} \pi_{i-1}(t) + \mu_{i+1} \pi_{i+1}(t) \\
\text{flows out} &\quad \text{flows in} \\
i &= 1, 2, \ldots
\end{align*}
\]

\[
\begin{align*}
\frac{d\pi_0(t)}{dt} &= -\lambda_0 \pi_0(t) + \mu_1 \pi_1(t) \\
\text{flow out} &\quad \text{flow in}
\end{align*}
\]
Example 1. Pure death process

\[
\begin{align*}
\lambda_i &= 0 & i = 0, 1, 2, \ldots \\
\mu_i &= i\mu
\end{align*}
\]

all individuals have the same mortality rate \( \mu \)

\[
\pi_i(0) = \begin{cases} 
1 & i = n \\
0 & i \neq n 
\end{cases}
\]

the system starts from state \( n \)

\[
\begin{array}{c}
0 \\
\mu
\end{array} \xrightarrow{1} \begin{array}{c}
1 \\
2\mu
\end{array} \xrightarrow{2} \begin{array}{c}
2 \\
3\mu
\end{array} \cdots \xrightarrow{(n-1)\mu} \begin{array}{c}
(n-1) \\
(n) \\
n\mu
\end{array}
\]

State 0 is an absorbing state, other states are transient

\[
\begin{align*}
\frac{d}{dt} \pi_n(t) &= -n\mu \pi_n(t) \\
\frac{d}{dt} \pi_i(t) &= (i + 1)\mu \pi_{i+1}(t) - i\mu \pi_i(t) & i = 0, 1, \ldots, n - 1
\end{align*}
\]

\[
\pi_i(t) = (i + 1)e^{-i\mu t} \int_0^t \pi_{i+1}(t')e^{i\mu t'} dt' \\
\pi_{n-1}(t) = ne^{-(n-1)\mu t} \int_0^t e^{-n\mu t'} \frac{e^{(n-1)\mu t'}}{e^{-\mu t'}} dt' = n e^{-(n-1)\mu t}(1 - e^{-\mu t})
\]

Recursively \[
\pi_i(t) = \binom{n}{i}(e^{-\mu t})^i(1 - e^{-\mu t})^{n-i}
\]

Binomial distribution: the survival probability at time \( t \) is \( e^{-\mu t} \) independent of others
Example 2. Pure birth process (Poisson process)

\[
\begin{aligned}
  \lambda_i &= \lambda & i &= 0, 1, 2, \ldots \\
  \mu_i &= 0 & i &= 0, 1, 2, \ldots \\
  \pi_i(0) &= \begin{cases} 
  1 & i = 0 \\
  0 & i > 0 
  \end{cases}
\end{aligned}
\]

birth probability per time unit is constant \(\lambda\)

initially the population size is 0

All states are transient

\[
\begin{aligned}
  \frac{d}{dt} \pi_i(t) &= -\lambda \pi_i(t) + \lambda \pi_{i-1}(t) & i &> 0 \\
  \frac{d}{dt} \pi_0(t) &= -\lambda \pi_0(t)
\end{aligned}
\]

\[
\Rightarrow \quad \pi_0(t) = e^{-\lambda t}
\]

\[
\frac{d}{dt}(e^{\lambda t} \pi_i(t)) = \lambda \pi_{i-1}(t)e^{\lambda t} \quad \Rightarrow \quad \pi_i(t) = e^{-\lambda t} \lambda \int_0^t \pi_{i-1}(t')e^{\lambda t'}dt'
\]

\[
\pi_1(t) = e^{-\lambda t} \lambda \int_0^t e^{-\lambda t'} e^{\lambda t'} dt' = e^{-\lambda t}(\lambda t)
\]

Recursively

\[
\pi_i(t) = \frac{(\lambda t)^i}{i!} e^{-\lambda t}
\]

Number of births in interval \((0, t) \sim \text{Poisson}(\lambda t)\)
Example 3. A single server system

- constant arrival rate $\lambda$ (Poisson arrivals)
- stopping rate of the service $\mu$ (exponential distribution)

The states of the system

\[
\begin{align*}
\begin{array}{c}
0 \\
1
\end{array}
\end{align*}
\]

\[\begin{array}{c}
\text{Exp}(\mu) \quad \text{Exp}(\lambda)
\end{array}\]

\[\begin{align*}
\frac{d}{dt} \pi_0(t) &= -\lambda \pi_0(t) + \mu \pi_1(t) \\
\frac{d}{dt} \pi_1(t) &= \lambda \pi_0(t) - \mu \pi_1(t)
\end{align*}\]

\[Q = \begin{pmatrix}
-\lambda & \lambda \\
\mu & -\mu
\end{pmatrix}\]

BY adding both sides of the equations

\[\frac{d}{dt}(\pi_0(t) + \pi_1(t)) = 0 \quad \Rightarrow \quad \pi_0(t) + \pi_1(t) = \text{constant} = 1 \quad \Rightarrow \quad \pi_1(t) = 1 - \pi_0(t)\]

\[\frac{d}{dt}\pi_0(t) + (\lambda + \mu)\pi_0(t) = \mu \quad \Rightarrow \quad \frac{d}{dt}(e^{(\lambda+\mu)t}\pi_0(t)) = \mu e^{(\lambda+\mu)t}\]

\[\pi_0(t) = \frac{\mu}{\lambda+\mu} + (\pi_0(0) - \frac{\mu}{\lambda+\mu})e^{-(\lambda+\mu)t}\]

\[\pi_1(t) = \frac{\lambda}{\lambda+\mu} + (\pi_1(0) - \frac{\lambda}{\lambda+\mu})e^{-(\lambda+\mu)t}\]

\text{equilibrium distribution} \quad \text{deviation from the equilibrium decays exponentially}
Summary of the analysis on Markov processes

1. Find the state description of the system
   - no ready recipe
   - often an appropriate description is obvious
   - sometimes requires more thinking
   - a system may Markovian with respect to one state description but non-Markovian with respect to another one (the state information is not sufficient)
   - finding the state description is the creative part of the problem

2. Determine the state transition rates
   - a straight forward task when holding times and interarrival times are exponential

3. Solve the balance equations
   - in principle straight forward (solution of a set of linear equations)
   - the number of unknowns (number of states) can be very great
   - often the special structure of the transition diagram can be exploited
Global balance

\[ \sum_{j \neq i} \pi_j q_{j,i} = \sum_{j \neq i} \pi_i q_{i,j} \]

one equation per each state

\[ i = 0, 1, \ldots, n \]

\[ \pi \cdot Q = 0 \]

one equation is redundant

\[ \pi_0 + \pi_1 + \cdots + \pi_n = 1 \]

normalization condition
Example 1. A queueing system

The number of customers in system $N$ is an appropriate state variable

- uniquely determines the number of customers in service and in waiting room
- after each arrival and departure the remaining service times
  of the customers in service are $\text{Exp}(\mu)$ distributed (memoryless)
Example 2. Call blocking in an ATM network

A virtual path (VP) of an ATM network is offered calls of two different types.

\[
\begin{align*}
R_1 &= 1\text{Mbps} \\
\lambda_1 &= \text{arrival rate} \\
\mu_1 &= \text{mean holding time} \\
R_2 &= 2\text{Mbps} \\
\lambda_2 &= \text{arrival rate} \\
\mu_2 &= \text{mean holding time}
\end{align*}
\]

a) The capacity of the link is large (infinite)
Call blocking in an ATM network (continued)

b) The capacity of the link is 4.5 Mbps