D12/1 Consider the following network with 4 nodes and 10 links. The set of nodes is denoted by $\mathcal{N} = \{a, b, c, d\}$, and the set of links by $\mathcal{J} = \{1, 2, \ldots, 10\}$. The properties of various links are given in the table below (j = link index, n_j = origin node, m_j = destination node, c_j = link capacity).

j	1	2	3	4	5	6	7	8	9	10
n_j	a	b	a	c	a	d	b	c	c	d
m_j	b	a	c	a	d	a	c	b	d	c
c_j	10	10	10	10	10	10	4	4	4	4

(a) Draw the network topology.

(b) What is the number of OD pairs?

(c) What is the total number of paths?

(d) What is the total number of shortest paths assumed that the links have unit weights ($w_j = 1$ for all j)?

D12/2 Consider again the network specified in the previous problem. The network is loaded by the traffic demands given by the traffic matrix T,

$$\mathbf{T} = \begin{pmatrix} 0 & 5 & 5 & 5 \\ 5 & 0 & 2 & 2 \\ 5 & 2 & 0 & 2 \\ 5 & 2 & 2 & 0 \end{pmatrix}.$$

(a) Load Balancing Problem refers to the minimization of the maximum relative link load. Give the optimal solution with confirming arguments.

(b) Determine the relative link loads resulting from this optimal routing scheme.

D12/3 Consider still the network and traffic specified in the previous problems. Assume now that the shortest path alogorithm with unit weights ($w_j = 1$ for all j) is applied (instead of optimal routing) together with the ECMP principle.

(a) Determine the relative link loads resulting from this shortest path routing routing scheme.

(b) Give a better routing scheme achieved by modifying the link weights.

D12/1 (a) Figure 1

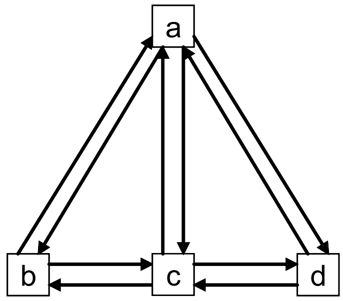

Figure 1: [D12/1] Topology.

(b) Since all the four nodes are connected, the number of OD-pairs is equal to $4 \cdot 3 = 12$.

(c,d) The table below gives the number of paths and the number of shortest paths for all OD-pairs. Totally there are 38 paths and 14 shortest paths in the whole network. For example, for the OD-pair (b,d) there are two shortest paths, namely b $\to$ a $\to$ d and b $\to$ c $\to$ d, both of length two hops.

k	OD-pair	nr of paths	nr of shortest paths
1	(a,b)	3	1
2	(a,c)	3	1
3	(a,d)	3	1
4	(b,a)	3	1
5	(b,c)	3	1
6	(b,d)	4	2
7	(c,a)	3	1
8	(c,b)	3	1
9	(c,d)	3	1
10	(d,a)	3	1
11	(d,b)	4	2
12	(d,c)	3	1
	totally	38	14

D12/2 (a) Objective: Minimize the maximum relative link load, see L12/23 and L12/25.

Let us start with the relative link loads ρ_j resulting from the shortest path routes (with the least number of hops). The traffic to and from node a uses clearly one-hop paths. Due to the symmetric traffic and link capacities, for all links $a \to \cdot$ and $\cdot \to a$, we have

$$\rho_j = \frac{y_j}{c_j} = \frac{5}{10} = \frac{1}{2} = 0.50$$

Similarly, the OD-pairs (b,c), (c,b), (c,d), and (d,c) utilize one-hop paths. Due to the symmetric traffic and link capacities, the relative link loads for the corresponding links j read as follows:

$$\rho_j = \frac{y_j}{c_j} = \frac{2}{4} = \frac{1}{2} = 0.50$$

We still need to consider the traffic of OD-pairs (b,d) and (d,b). For the OD-pair (b,d), there are two minimum-hop paths: b $\rightarrow$ a $\rightarrow$ d and b $\rightarrow$ c $\rightarrow$ d. Let ϕ denote the fraction of this traffic that uses path b $\rightarrow$ a $\rightarrow$ d. The remaining fraction $1 - \phi$ uses the other minimum-hop path b $\rightarrow$ c $\rightarrow$ d. As a function of ϕ, the total loads of links (b,a) and (b,c) will be

$$\rho_{\text{ba}} = \frac{y_{\text{ba}}}{c_{\text{ba}}} = \frac{5 + \phi \cdot 2}{10}, \quad \rho_{\text{bc}} = \frac{y_{\text{bc}}}{c_{\text{ba}}} = \frac{2 + (1-\phi) \cdot 2}{4} = \frac{2 - \phi}{2}$$

The maximum of these relative link loads is minimized when they are equal:

$$\rho^*_{\text{ba}} = \rho^*_{\text{bc}} \Leftrightarrow \frac{5 + \phi^* \cdot 2}{10} = \frac{2 - \phi^*}{2} \Leftrightarrow \phi^* = \frac{5}{7} = 0.714$$

Thus, it is optimal that fraction $\phi^* = 5/7 = 0.714$ of the OD-pair (b,d) traffic uses path b $\rightarrow$ a $\rightarrow$ d. By symmetry, we finally deduce that it is optimal that fraction $\phi^* = 5/7 = 0.714$ of the opposite OD-pair (d,b) traffic uses path d $\rightarrow$ a $\rightarrow$ b.

(b) As mentioned in (a), the optimal routing results in equal relative loads on links (b,a) and (b,c):

$$\rho^*_{\text{ba}} = \rho^*_{\text{bc}} = \frac{5 + \frac{5}{7} \cdot 2}{10} = \frac{45}{70} = \frac{9}{14} = 0.64$$

By symmetry, we deduce that

$$\rho^*_{\text{ba}} = \rho^*_{\text{bc}} = \rho^*_{\text{ad}} = \rho^*_{\text{cd}} = \rho^*_{\text{da}} = \rho^*_{\text{dc}} = \rho^*_{\text{ab}} = \rho^*_{\text{cb}} = \frac{9}{14} = 0.64$$

In addition, the remaining relative link loads are as follows:

$$\rho^*_{\text{ac}} = \rho^*_{\text{ca}} = \frac{5}{10} = 0.50$$

The minimum value for the maximum relative link load is thus $9/14 = 0.64$. All the relative link loads that result from the optimal routing are presented in Figure 2 (left-hand-side).

Note that there is another routing that leads to the same minimum value for the maximum relative link load. In this alternative routing scheme, a fraction ϕ^* of the OD-pair (b,d) traffic uses path b $\rightarrow$ c $\rightarrow$ a $\rightarrow$ d, while the remaining fraction $1 - \phi^*$ uses the path b $\rightarrow$ a $\rightarrow$ c $\rightarrow$ d. However, this alternative solution uses longer paths, and thus optimizes only the load balancing problem presented in L12/23 (but not the finer-grained problem presented in L12/25).

D12/3 (a) Assume now that the shortest path alogorithm with unit weights ($w_j = 1$ for all j) is applied together with the ECMP principle (L12/18). As a result, all the other OD-pairs but (b,d) and (d,b) use corresponding direct links j. As we saw in D12/2(a), this results in equal relative loads for these links j,

$$\rho_j = \frac{1}{2} = 0.50$$

Consider then the OD-pair (b,d) traffic. Let then ϕ° denote the fraction of this traffic that uses path the two-hop path b → a → d. The remaining fraction $1-\phi$ uses the other two-hop path b → c → d. Due to the ECMP principle, we deduce that

$$\phi^\circ = \frac{1}{2} = 0.50$$

Thus, the total relative link loads for the links along these routes become as follows:

$$\rho^\circ_{\text{ba}} = \rho^\circ_{\text{ad}} = \frac{5+\phi^\circ \cdot 2}{10} = \frac{5+\frac{1}{2}\cdot 2}{10} = \frac{3}{5} = 0.60,$$
$$\rho^\circ_{\text{bc}} = \rho^\circ_{\text{cd}} = \frac{2+(1-\phi^\circ)\cdot 2}{4} = \frac{2+\frac{1}{2}\cdot 2}{4} = \frac{3}{4} = 0.75$$

By symmetry,

$$\rho^\circ_{\text{da}} = \rho^\circ_{\text{ab}} = \frac{3}{5} = 0.60, \quad \rho^\circ_{\text{dc}} = \rho^\circ_{\text{cb}} = \frac{3}{4} = 0.75$$

For the remaining links, we clearly have

$$\rho^\circ_{\text{ac}} = \rho^\circ_{\text{ca}} = \frac{1}{2} = 0.50$$

The shortest path alogorithm with unit weights combined with ECMP principle results tus in the maximum relative link load of $3/4 = 0.75$ (which is clearly greater than optimal value $9/14 = 0.64$). All the relative link loads that result from this routing are presented in Figure 2 (middle).

(b) By changing the link weights w_j, we try to decrease the relative load $3/4 = 0.75$ of the bottleneck links (b,c), (c,d), (d,c), and (c,b). By choosing

$$\tilde{w}_{\text{ab}} = \tilde{w}_{\text{ba}} = \tilde{w}_{\text{ac}} = \tilde{w}_{\text{ca}} = \tilde{w}_{\text{ad}} = \tilde{w}_{\text{da}} = 2$$
$$\tilde{w}_{\text{bc}} = \tilde{w}_{\text{cb}} = \tilde{w}_{\text{cd}} = \tilde{w}_{\text{dc}} = 3$$

we change the shortest paths for OD-pairs (b,d) and (d,b) but not for any other OD-pairs. With these modified weights, the unique shortest paths for these OD-pairs (b,d) and (d,b) will be b → a → d and d → a → b, respectively. As a result, the relative link loads are changed to

$$\tilde{\rho}_{\text{ba}} = \tilde{\rho}_{\text{ad}} = \tilde{\rho}_{\text{da}} = \tilde{\rho}_{\text{ab}} = \frac{5+2}{10} = \frac{7}{10} = 0.70$$
$$\tilde{\rho}_{\text{ac}} = \tilde{\rho}_{\text{ca}} = \frac{5}{10} = \frac{1}{2} = 0.50$$
$$\tilde{\rho}_{\text{bc}} = \tilde{\rho}_{\text{cd}} = \tilde{\rho}_{\text{dc}} = \tilde{\rho}_{\text{cb}} = \frac{2}{4} = \frac{1}{2} = 0.50$$

Now we see that maximum relative link load is reduced to $7/10 = 0.70$. In fact, this is the best result that can be achieved by modifying the link weights. All the relative link loads that result from these modified link weights are presented in Figure 2 (right-hand-side).

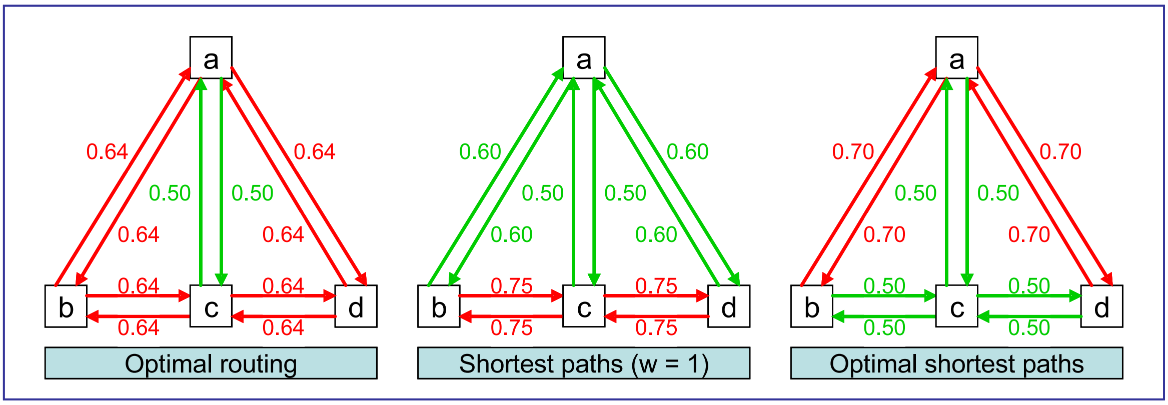

Figure 2: [D12/2,D12/3] Relative link loads for various routing schemes. *Left*: Optimal routing. *Middle:* Shortest paths with unit weights. *Right*: Shortest paths with optimized link weights.