D11/1 Generate four (pseudo) random numbers from the U$(0,1)$ distribution using the MCG algorithm presented in the lecture with parameters $m = 2^{31} - 1$, $a = 16\,807$, and $Z_0 = 920\,107$.

D11/2 Utilizing the random numbers generated in the previous problem, generate four random numbers from each of the following distributions:

(a) U$(1,2)$,

(b) Geom$(1/2)$, and

(c) Exp(2).

D11/3 Simulation runs have yielded the following independent observations X_i for a performance parameter α: 2.47, 5.32, 3.63, 4.16, 2.40, and 6.07. Determine the 95% confidence interval for α assuming that the variance is known, $D^2[X_i] = 2$.

D11/4 Simulate, according to the discrete event simulation principles, the evolution of the queue length process $Q(t)$ of an M/M/1-FIFO queue (with parameters $\lambda = 1/2$ and $\mu = 1$) during the interval $[0,T]$, where $T = 2000$. Assume that the system is empty in the beginning ($Q(0) = 0$). Make $n = 100$ independent simulation runs. (Independent means that the seed value for the random number generation changes.) In each simulation run, determine the mean queue length X in the interval $[T_0, T]$, where $T_0 = 1000$, from the equation

$$X = \frac{1}{T - T_0} \int_{T_0}^{T} Q(t)dt.$$

By this way, you get n IID samples $X_1, X_2, \ldots, X_n$ of the mean queue length in this interval.

(a) Calculate and plot the sample average $\bar{X}_m$, for $m = 10, 20, \ldots, n$,

$$\bar{X}_m = \frac{1}{m} \sum_{i=1}^{m} X_i.$$

(b) Calculate and plot the square root of the sample variance, S_m, for $m = 10, 20, \ldots, n$,

$$S_m = \sqrt{\frac{1}{m-1} \sum_{i=1}^{m} (X_i - \bar{X}_m)^2}.$$

(c) Calculate and plot the confidence interval for the sample average $\bar{X}_m$ at confidence level 95% for $m = 10, 20, \ldots, n$, assuming that the samples are IID and from a normal distribution, but with an unknown variance.

D11/1 The MCG algorithm (L11/25):

$$Z_{i+1} = (aZ_i) \mod m$$

With $m = 2^{31} - 1 = 2\,147\,483\,647$, $a = 16\,807$, and $Z_0 = 920\,107$, we get

$$Z_1 = \ 431\,852\,820, \quad Z_2 = 1\,803\,102\,527, \quad Z_3 = 1\,602\,428\,472, \quad Z_4 = \ 422\,911\,877$$

By a normalization (L11/26),

$$U_i = \frac{Z_i}{m},$$

these become as random numbers generated from the $\mathrm{U}(0,1)$ distribution:

$$U_1 = 0.201097, \quad U_2 = 0.839635, \quad U_3 = 0.746189, \quad U_4 = 0.196934$$

D11/2 (a) By rescaling (L11/27),

$$X_i = 1 + U_i,$$

the random numbers U_i of the previous problem become as random numbers generated from the $\mathrm{U}(1,2)$ distribution:

$$X_1 = 1.20110, \quad X_2 = 1.83964, \quad X_3 = 1.74619, \quad X_4 = 1.19693$$

(b) Let us start with the cumulative distribution function of the $\mathrm{Geom}(p)$ distribution (L4/30):

$$F(n) = P\{X \le n\} = \sum_{i=0}^{n} P\{X = i\} = (1-p)\sum_{i=0}^{n} p^i = 1 - p^{n+1}, \quad n = 0, 1, \ldots$$

For $p = 1/2$, we get

$$F(0) = 1/2 = 0.500, \quad F(1) = 3/4 = 0.750, \quad F(2) = 7/8 = 0.875, \quad \ldots$$

The discretization method (L11/28),

$$X_i = \min\{n \mid F(n) \ge U_i\},$$

results in the following random numbers generated from the $\mathrm{Geom}(1/2)$ distribution:

$$X_1 = 0, \quad X_2 = 2, \quad X_3 = 1, \quad X_4 = 0$$

(c) The inverse transform method (L11/30),

$$X_i = -\frac{1}{2}\log(U_i),$$

results in the following random numbers generated from the $\mathrm{Exp}(2)$ distribution:

$$X_1 = 0.801984, \quad X_2 = 0.087394, \quad X_3 = 0.146388, \quad X_4 = 0.812444$$

D11/3 The sample size is $n = 6$, and the sample mean equals $\bar{X}_n = 4.01$. In addition, the variance of a single sample is known, $\sigma^2 = D^2[X_i] = 2$. Thus, the 95% confidence interval for α is (L11/43)

$$\bar{X}_n \pm z_{0.975} \cdot \frac{\sigma}{\sqrt{n}} = 4.01 \pm 1.96 \cdot \frac{\sqrt{2}}{\sqrt{6}} = 4.01 \pm 1.13 = (2.88, 5.14)$$

D11/4 The simulator is designed in such a way that any arrival and service time distribution is acceptable. Because we are interested in the mean queue length, we do not have to gather information on the time spent by each individual user etc. The information that we need to gather is:

- The number of customers in the system at each instant,
- information on the time instant of the next arrival,
- information on the time instant of the next departure, and
- the cumulative sum of $Q(t)$.

The basic idea is the following:

1. Initialization of variables, generation of the initial state etc.
2. Has the time interval (T) been reached? If no, go to step 3. If yes, go to step 6.
3. If the next event is an arrival: Increment the state (number of customers in the system) variable by one. Update the cumulative sum of $Q(t)$ and calculate the time instant of the next arrival. If the system was empty at the time instant of the arrival, calculate the next departure instant (when an arriving customer enters an empty system, the service starts immediately).
4. If the next event is a departure: Decrement the state variable by one. Update the cumulative sum of $Q(t)$ and calculate the next time instant for a departure. If the system is emptied, set the time for the next departure to infinity.
5. Go to step 2.
6. Calculate the required statistics.

In this exercise we wish to study the convergence of the estimate, as the number of observations increases. Fig. 1 (left-hand-side) shows how the mean $\bar{X}_n$ of the queue length stabilizes as the sample size grows $(10, \ldots, 100)$. The left-hand-side also includes the 95% confidence interval. The confidence interval becomes narrower as the sample size grows. The width of the confidence interval is relative to the standard deviation $D[\bar{X}_m]$, which decreases with increasing m (L11/40),

$$D[\bar{X}_m] = \frac{D[X_1]}{\sqrt{m}}.$$

Here $D[X_1]$ refers to the (unknown) standard deviation of a single sample tha can be estimated by the sample standard deviation S_m (L11/45),

$$S_m = \sqrt{\frac{1}{m-1} \sum_{i=1}^{m} (X_i - \bar{X}_m)^2}$$

In addition, the estimate converges to theoretical value ($\rho/(1-\rho) = 1$ in this case), since the effect of initial transient is compencated by collecting the statistics just after the warm-up phase (which ends at $T_0 = 1000$).

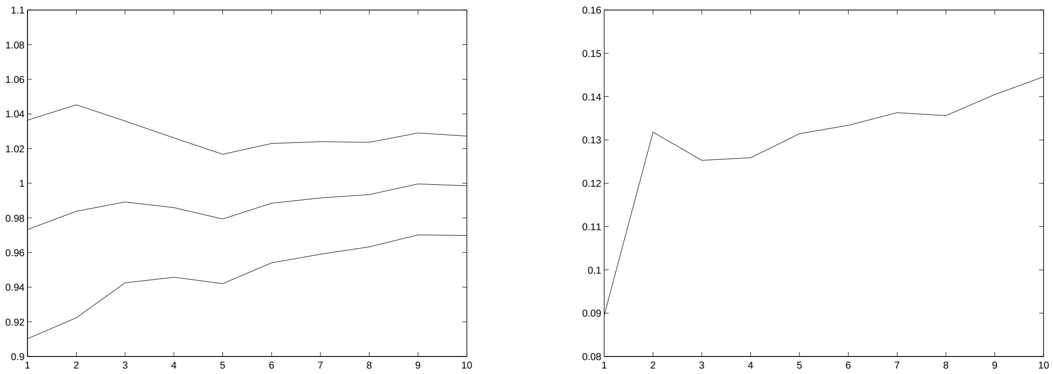

Figure 1: [D11/4] Left: Sample mean and its 95% confidence interval. Right: Sample standard deviation. Both as a function of the sample size.