TKK HELSINKI UNIVERSITY OF TECHNOLOGY **Demonstrations**
Department of Communications and Networking **Lecture 8**
S-38.1145 Introduction to Teletraffic Theory, Spring 2008 8.2.2008

D8/1 Consider the following simple teletraffic model with a single server ($n = 1$): Customers arrive according to a Poisson process with intensity λ. Service times are IID and exponentially distributed with mean $1/\mu$. The number of waiting places is finite ($0 < m < \infty$). Queueing discipline is FIFO. Let $X(t)$ denote the number of customers in the system at time t, which is a Markov process.

(a) What is the traffic model in question (with Kendall's notation)?

(b) Draw the state transition diagram of $X(t)$.

(c) Derive the equilibrium distribution of $X(t)$.

(d) What is the probability that an arriving customer is lost?

(e) What is the probability that an arriving customer that is not lost has to wait?

D8/2 Consider the M/M/2/3 model with mean customer interarrival time of $1/\lambda$ time units and mean service time of $1/\mu$ time units. Let $X(t)$ denote the number of customers in the system at time t, which is a Markov process.

(a) Draw the state transition diagram of $X(t)$.

(b) Derive the equilibrium distribution of $X(t)$.

(c) Assume that $\lambda = \mu$. What is the probability that an arriving customer is lost?

(d) Assume again that $\lambda = \mu$. What is the utilization factor of the system, that is, the mean number of busy servers divided by the total number of servers?

D8/3 Consider a (lossy) queueing system with two servers and one waiting place. As in D7/3, customers arrive in independent batches of size 1 or 2. Both sizes are equally probable. These batches arrive according to a Poisson process with intensity λ. The whole batch is lost whenever the system is full at the arrival time. But if the waiting place is free when a new batch of size 2 arrives, only one of the arriving customers is lost. The customers are served individually and independently with the service time following the Exp(μ) distribution. Let $X(t)$ denote the number of customers in the system at time t, which is a Markov process.

(a) Draw the state transition diagram of $X(t)$.

(b) Derive the equilibrium distribution of $X(t)$.

(c) Assume that $\lambda = \mu$. What is the utilization factor of the system, that is, the mean number of busy servers divided by the total number of servers?

(d) Assume $1/\lambda = 1/\mu = 1$ (time unit). What is the mean waiting time for the customers that are not lost?

D8/1 (a) This is the M/M/1/N model where $N = m + 1$.

(b) Figure 1.

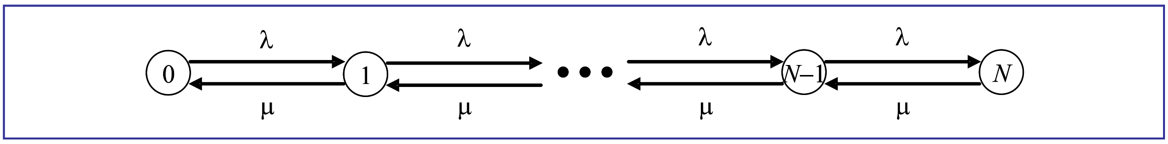

Figure 1: [D8/1] State transiotion diagram.

(c) We see from Figure 1 that $X(t)$ is an irreducible birth-death-process (L6/16). Since the state space is finite, the equilibrium distribution π exists, and it can be derived based on the local balance equations (LBE) and the normalization condition (N), cf. L6/17.

Let us start with the LBE's for states $i-1$ and i, where $i = 1, \ldots, N$:

$$\pi_{i-1}\lambda = \pi_i \mu$$

This results in the following recursion:

$$\pi_i = \pi_{i-1}\,\frac{\lambda}{\mu} = \pi_{i-2}(\frac{\lambda}{\mu})^2 = \ldots = \pi_0(\frac{\lambda}{\mu})^i = \pi_0\,\rho^i$$

where $\rho = \lambda/\mu$. The remaining probability π_0 is determined by (N):

$$\pi_0 + \pi_1 + \ldots + \pi_N = \pi_0 \sum_{i=0}^{N} \rho^i = 1$$

so that

$$\pi_0 = \begin{cases} \dfrac{1}{N+1}, & \text{if } \rho = 1, \\ \dfrac{1-\rho}{1-\rho^{N+1}}, & \text{otherwise.} \end{cases}$$

Thus, the equilibrium distribution is the truncated geometric distribution:

$$\pi_i = \begin{cases} \dfrac{1}{N+1}, & \text{if } \rho = 1, \\ \dfrac{(1-\rho)\rho^i}{1-\rho^{N+1}}, & \text{otherwise.} \end{cases} \qquad (i = 0, 1, \ldots, N)$$

(d) A customer is lost if the system is full at the arrival time. Due to the PASTA property (L5/28) of the Poisson arrival processs, this happens with probability which is equal to the equilibrium probability π_N. Thus,

$$P\{\text{“a customer is lost”}\} = P\{X = N\} = \pi_N = \begin{cases} \dfrac{1}{N+1}, & \text{if } \rho = 1, \\ \dfrac{(1-\rho)\rho^N}{1-\rho^{N+1}}, & \text{otherwise.} \end{cases}$$

(e) A customer that is not lost will get an immediate service (without any waiting) if the system is empty at the arrival time. Again, we apply the PASTA property (L5/28) to get the required probabilities as follows:

$$
\begin{aligned}
&P\{\text{“a customer has to wait”} \mid \text{“not lost”}\} \\
&= 1 - P\{X = 0 \mid X < N\} \\
&= 1 - \frac{P\{X = 0\}}{P\{X < N\}} \\
&= 1 - \frac{\pi_0}{1 - \pi_N} \\
&= \begin{cases} \dfrac{N-1}{N}, & \text{if } \rho = 1, \\ \dfrac{\rho - \rho^N}{1 - \rho^N}, & \text{otherwise.} \end{cases}
\end{aligned}
$$

D8/2 (a) Figure 2.

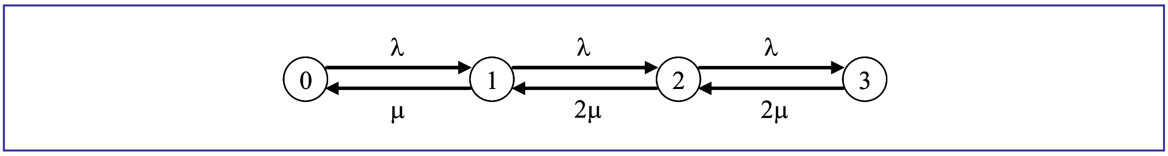

Figure 2: [D8/2] State transition diagram.

(c) We see from Figure 1 that $X(t)$ is an irreducible birth-death-process (L6/16). Since the state space is finite, the equilibrium distribution π exists, and it can be derived based on the local balance equations (LBE) and the normalization condition (N), cf. L6/17.
Let us start with the LBE's for states $i-1$ and i, where $i = 1, 2, 3$:

$$\pi_0 \lambda = \pi_1 \mu, \quad \pi_1 \lambda = \pi_2\, 2\mu, \quad \pi_2 \lambda = \pi_3\, 2\mu$$

The other probabilities are now solved as a function of π_0:

$$\pi_1 = \pi_0\, \frac{\lambda}{\mu}, \quad \pi_2 = \pi_0\, \frac{1}{2}(\frac{\lambda}{\mu})^2, \quad \pi_3 = \pi_0\, \frac{1}{4}(\frac{\lambda}{\mu})^3$$

The remaining probability π_0 is determined by (N):

$$\pi_0 + \pi_1 + \ldots + \pi_N = \pi_0 \left(1 + \frac{\lambda}{\mu} + \frac{1}{2}(\frac{\lambda}{\mu})^2 + \frac{1}{4}(\frac{\lambda}{\mu})^3\right) = 1$$

Thus, the equilibrium distribution is

$$
\begin{aligned}
\pi_0 &= \frac{1}{1 + \frac{\lambda}{\mu} + \frac{1}{2}(\frac{\lambda}{\mu})^2 + \frac{1}{4}(\frac{\lambda}{\mu})^3}, \quad
\pi_1 = \frac{\frac{\lambda}{\mu}}{1 + \frac{\lambda}{\mu} + \frac{1}{2}(\frac{\lambda}{\mu})^2 + \frac{1}{4}(\frac{\lambda}{\mu})^3}, \\
\pi_2 &= \frac{\frac{1}{2}(\frac{\lambda}{\mu})^2}{1 + \frac{\lambda}{\mu} + \frac{1}{2}(\frac{\lambda}{\mu})^2 + \frac{1}{4}(\frac{\lambda}{\mu})^3}, \quad
\pi_3 = \frac{\frac{1}{4}(\frac{\lambda}{\mu})^3}{1 + \frac{\lambda}{\mu} + \frac{1}{2}(\frac{\lambda}{\mu})^2 + \frac{1}{4}(\frac{\lambda}{\mu})^3}
\end{aligned}
$$

For (c) and (d), we compute the equilibrium distribution under the assumption that $\lambda = \mu$:

$$\pi_0 = \frac{4}{11} = 0.36, \quad \pi_1 = \frac{4}{11} = 0.36, \quad \pi_2 = \frac{2}{11} = 0.18, \quad \pi_3 = \frac{1}{11} = 0.09$$

(c) A customer is lost if the system is full at the arrival time. Due to the PASTA property (L5/28) of the Poisson arrival processs, this happens with probability which is equal to the equilibrium probability π_3. Thus,

$$P\{\text{“a customer is lost”}\} = \pi_3 = \frac{1}{11} = 0.09$$

(d) The mean number of busy servers is

$$E[X_{\mathrm{S}}] = \pi_1 + 2(\pi_2 + \pi_3) = \frac{4}{11} + 2 \cdot \left(\frac{2}{11} + \frac{1}{11}\right) = \frac{10}{11} = 0.91$$

Thus, the utilization factor becomes

$$E[U] = \frac{E[X_{\mathrm{S}}]}{n} = \frac{\frac{10}{11}}{2} = \frac{5}{11} = 0.45$$

D8/3 (a) Figure 3.

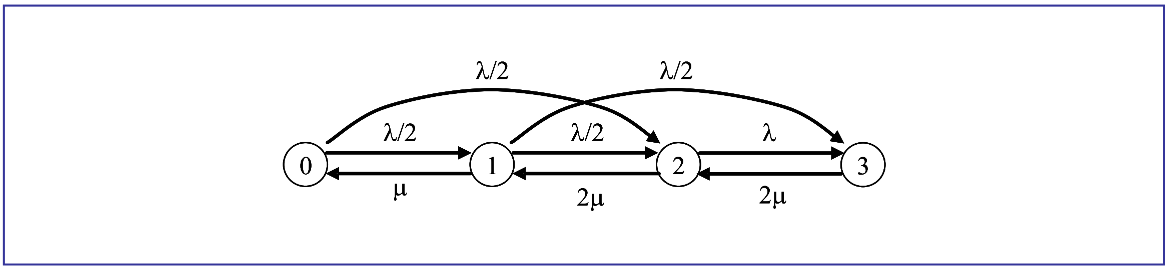

Figure 3: [D8/3] State transition diagram.

(b) We see from Figure 3 that the Markov process $X(t)$ is irreducible (L6/10). Since the state space is finite, the equilibrium distribution π exists, and it can be derived based on the global balance equations (GBE) and the normalization condition (N), cf. L6/11.

Let us start with the GBE's for states 0, 1, and 2:

$$\pi_0\lambda = \pi_1\mu, \quad \pi_1(\lambda + \mu) = \pi_0 \frac{\lambda}{2} + \pi_2\, 2\mu, \quad \pi_2(\lambda + 2\mu) = \pi_1 \frac{\lambda}{2} + \pi_3\, 2\mu$$

The other probabilities are now solved as a function of π_0:

$$\pi_1 = \pi_0 \frac{\lambda}{\mu}, \quad \pi_2 = \pi_0 \left(\frac{1}{4}(\frac{\lambda}{\mu}) + \frac{1}{2}(\frac{\lambda}{\mu})^2\right), \quad \pi_3 = \pi_0 \left(\frac{3}{8}(\frac{\lambda}{\mu})^2 + \frac{1}{4}(\frac{\lambda}{\mu})^3\right)$$

The remaining probability π_0 is determined by (N):

$$\pi_0 + \pi_1 + \pi_2 = \pi_0 \left(1 + \frac{5}{4}(\frac{\lambda}{\mu}) + \frac{7}{8}(\frac{\lambda}{\mu})^2 + \frac{1}{4}(\frac{\lambda}{\mu})^3\right) = 1,$$

so that the equilibrium distribution is

$$\pi_0 = \frac{1}{1+\frac{5}{4}(\frac{\lambda}{\mu})+\frac{7}{8}(\frac{\lambda}{\mu})^2+\frac{1}{4}(\frac{\lambda}{\mu})^3}, \quad \pi_1 = \frac{\frac{\lambda}{\mu}}{1+\frac{5}{4}(\frac{\lambda}{\mu})+\frac{7}{8}(\frac{\lambda}{\mu})^2+\frac{1}{4}(\frac{\lambda}{\mu})^3},$$

$$\pi_2 = \frac{\frac{1}{4}(\frac{\lambda}{\mu})+\frac{1}{2}(\frac{\lambda}{\mu})^2}{1+\frac{5}{4}(\frac{\lambda}{\mu})+\frac{7}{8}(\frac{\lambda}{\mu})^2+\frac{1}{4}(\frac{\lambda}{\mu})^3}, \quad \pi_3 = \frac{\frac{3}{8}(\frac{\lambda}{\mu})^2+\frac{1}{4}(\frac{\lambda}{\mu})^3}{1+\frac{5}{4}(\frac{\lambda}{\mu})+\frac{7}{8}(\frac{\lambda}{\mu})^2+\frac{1}{4}(\frac{\lambda}{\mu})^3}$$

For (c) and (d), we compute the equilibrium distribution under the assumption that $\lambda = \mu$:

$$\pi_0 = \frac{8}{27} = 0.30, \quad \pi_1 = \frac{8}{27} = 0.30, \quad \pi_2 = \frac{6}{27} = 0.22, \quad \pi_3 = \frac{5}{27} = 0.19$$

(c) The mean number of busy servers is

$$E[X_{\mathrm{S}}] = \pi_1 + 2(\pi_2+\pi_3) = \frac{8}{27} + 2\cdot\left(\frac{6}{27}+\frac{5}{27}\right) = \frac{10}{9} = 1.11$$

Thus, the utilization factor becomes

$$E[U] = \frac{E[X_{\mathrm{S}}]}{n} = \frac{\frac{10}{9}}{2} = \frac{5}{9} = 0.56$$

(d) There is a single customer waiting in state 3, while in all the other states no customers wait. Thus, the mean number of waiting customers is

$$E[X_{\mathrm{W}}] = \pi_3 = \frac{5}{27} = 0.19$$

New customers (that are not lost) enter the system with intensity

$$\lambda_{\mathrm{carried}} = \lambda(1-B_{\mathrm{C}}),$$

wher B_{C} is the (call blocking) probability that an arriving customer is lost. When this intensity $\lambda_{\mathrm{carried}}$ is known, we can use Little's formula to calculate the required mean waiting time.

To calculate the call blocking probability, we need to know the mean number of customers in a batch, $E[A]$, and the mean number of lost customers in a batch, $E[L]$. The former one is clearly

$$E[A] = 1\cdot P\{A=1\} + 2\cdot P\{A=2\} = 1\cdot\frac{1}{2} + 2\cdot\frac{1}{2} = \frac{3}{2} = 1.50$$

On the other hand, due to the PASTA property of the Poisson process (L5/28), an arriving batch sees the system in equilibrium. Thus,

$$\begin{aligned} E[L] &= \pi_2\,(1\cdot P\{A=2\}) + \pi_3\,(1\cdot P\{A=1\}+2\cdot P\{A=2\}) \\ &= \pi_2 P\{A=2\} + \pi_3 E[A] = \frac{6}{27}\cdot\frac{1}{2} + \frac{5}{27}\cdot\frac{3}{2} = \frac{7}{18} = 0.39 \end{aligned}$$

The call blocking probability B_{C} is their ratio:

$$B_{\mathrm{C}} = \frac{E[L]}{E[A]} = \frac{\frac{7}{18}}{\frac{3}{2}} = \frac{7}{27} = 0.26,$$

so that, due to Little's formula (L1/31), the mean waiting time for the customers that are not lost is finally

$$E[W] = \frac{E[X_{\mathrm{w}}]}{\lambda_{\mathrm{carried}}} = \frac{E[X_{\mathrm{w}}]}{\lambda(1 - B_{\mathrm{c}})} = \frac{\frac{5}{27}}{1 - \frac{7}{27}} = \frac{1}{4} = 0.25 \text{ (time units)}$$