D7/1 Consider a link in a circuit switched trunk network. Denote by $n$ the number of parallel channels. Users generate new calls according to a Poisson process. The mean interarrival time between new calls is denoted by $t$, and the mean call holding time by $h$.

(a) What is the traffic model in question (with Kendall’s notation)?

(b) Determine the time blocking, the call blocking, and the traffic carried for $n = 2$, $t = 4$ min, and $h = 3$ min.

D7/2 Consider a link in a circuit switched access network. Denote by $n$ the number of parallel channels. There are $k$ on-off type users generating new calls when idle, with $k > n$. The mean idle time is denoted by $t$, and the mean call holding time by $h$.

(a) What is the traffic model in question (with Kendall’s notation)?

(b) Determine the time blocking, the call blocking, and the traffic carried for $n = 2$, $k = 4$, $t = 9$ min, and $h = 3$ min.

D7/3 Consider a pure loss system with two servers. Customers arrive in independent batches of size 1 or 2. Both sizes are equally probable. These batches arrive according to a Poisson process with intensity $\lambda$. The whole batch is lost whenever the system is full at the arrival time. But if exactly one of the servers is idle when a new batch of size 2 arrives, only one of the arriving customers is lost. The customers are served individually and independently with the service time following the $\text{Exp}(\mu)$ distribution. Let $X(t)$ denote the number of customers in the system at time $t$, which is a Markov process.

(a) Draw the state transition diagram of $X(t)$.

(b) Derive the equilibrium distribution of $X(t)$.

(c) Assume that $\lambda = \mu$. What is the utilization factor of the system, that is, the mean number of busy servers divided by the total number of servers?

(d) Assume again that $\lambda = \mu$. What is the “call” blocking probability, that is, the probability that an arriving customer is lost?
D7/1  (a) This is the M/G/\(n/n\) model, that is, the Erlang model with a general call holding time distribution (L7/15).

(b) Now the arrival rate is \(\lambda = 1/t = 1/4\) calls/min, and the traffic intensity \(a = \lambda h = h/t = 3/4 = 0.75\) erl. In the Erlang model, the call blocking \(B_C\) and the time blocking \(B_t\) are equal, and they can be calculated from the Erlang formula (L7/20):

\[
B_C = B_t = \frac{\frac{a^n}{n!}}{\sum_{j=0}^{n} \frac{a^j}{j!}} = \frac{\frac{\left(\frac{3}{4}\right)^n}{n!}}{1 + \frac{3}{4} + \frac{1}{2}\left(\frac{3}{4}\right)^2} = \frac{9}{32 + 24 + 9} = \frac{9}{65} = 0.14
\]

Thus, the traffic carried is

\[
a_{\text{carried}} = a(1 - B_C) = \frac{3}{4} \cdot \left(1 - \frac{9}{65}\right) = \frac{42}{65} = 0.65\text{ erl}
\]

On the other hand, by Little’s formula (L1/31), the traffic carried equals the mean number of customers in the system, \(E[X]\). Since the equilibrium distribution of the Erlang model is the following truncated Poisson distribution (L7/18),

\[
\pi_i = \frac{\frac{a^i}{i!}}{\sum_{j=0}^{n} \frac{a^j}{j!}}, \quad i = 0, 1, 2,
\]

the mean value \(E[X]\) becomes

\[
E[X] = \sum_{i=0}^{n} i \cdot \pi_i = \frac{\frac{3}{4}}{1 + \frac{3}{4} + \frac{1}{2}\left(\frac{3}{4}\right)^2} + 2 \cdot \frac{\frac{1}{2}\left(\frac{3}{4}\right)^2}{1 + \frac{3}{4} + \frac{1}{2}\left(\frac{3}{4}\right)^2}
\]

\[
= \frac{24}{32 + 24 + 9} + 2 \cdot \frac{9}{32 + 24 + 9} = \frac{42}{65} = 0.65,
\]

as it should be.

D7/2  (a) This is the M/G/\(n/n/k\) model, that is, the Engset model with a general call holding time distribution (L7/32).

(b) When idle, a user becomes active with intensity \(\nu = 1/t = 1/9\) times/min. Correspondingly, when active, a user becomes idle with intensity \(\mu = 1/h = 1/3\) times/min. Thus, \(\nu/\mu = 3/9 = 1/3 = 0.33\). The formula for the time blocking is given by (L7/36)

\[
B_t = \frac{\binom{k}{n} \left(\frac{\nu}{\mu}\right)^n}{\sum_{j=0}^{n} \binom{k}{j} \left(\frac{\nu}{\mu}\right)^j} = \frac{6\left(\frac{1}{3}\right)^2}{1 + 4\left(\frac{1}{3}\right) + 6\left(\frac{1}{3}\right)^2} = \frac{6}{9 + 12 + 6} = \frac{2}{9} = 0.22
\]

The call blocking equals the time blocking in a modified system with one less customer, and can be calculated using the Engset formula (L7/40):

\[
B_C = \frac{\binom{k-1}{n} \left(\frac{\nu}{\mu}\right)^n}{\sum_{j=0}^{n} \binom{k-1}{j} \left(\frac{\nu}{\mu}\right)^j} = \frac{3\left(\frac{1}{3}\right)^2}{1 + 3\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right)^2} = \frac{1}{3 + 3 + 1} = \frac{1}{7} = 0.14
\]

By Little’s formula (L1/31), the traffic carried equals the mean number of customers in the system, \(E[X]\). Since the equilibrium distribution of the Engset model is the following truncated binomial distribution (L7/35),

\[
\pi_i = \frac{\binom{k}{i} \left(\frac{\nu}{\mu}\right)^i}{\sum_{j=0}^{n} \binom{k}{j} \left(\frac{\nu}{\mu}\right)^j}, \quad i = 0, 1, 2,
\]
the mean value $E[X]$ becomes

$$E[X] = \sum_{i=0}^{n} i \cdot \pi_i = \frac{4(\frac{1}{4})}{1 + 4(\frac{1}{4}) + 6(\frac{1}{4})^2} + 2 \cdot \frac{6(\frac{1}{4})^2}{1 + 4(\frac{1}{4}) + 6(\frac{1}{4})^2}$$

$$= \frac{12}{9 + 12 + 6} + 2 \cdot \frac{6}{9 + 12 + 6} = \frac{8}{9} = 0.89$$

Thus, the traffic carried is $a_{\text{carried}} = E[X] = 0.89$ erl.

Note: Due to the finite population of the Engset model, there are two slightly different definitions for the offered traffic: the hypothetical offered traffic $a^h_{\text{offered}}$ and the realized offered traffic $a^r_{\text{offered}}$. According to L7/35, the (hypothetical) offered traffic is in this case

$$a^h_{\text{offered}} = \frac{kn}{\nu + \mu} = \frac{k(\frac{1}{2})}{1 + (\frac{\nu}{\mu})} = \frac{4(\frac{1}{4})}{1 + (\frac{1}{3})} = 1.$$ 

So this is equal to the carried traffic in the corresponding lossless system (that is, the binomial model M/G/k/k/k). As it is reasonable to require, this characterization of the offered traffic is independent of the system parameters (such as the number of servers, $n$). However, the realized offered traffic is different from this due to the feedback mechanism of the model: the same customers return to the system (after an idle period). The (average) realized arrival rate is clearly

$$\lambda^r = \sum_{i=0}^{n} (k - i) \nu \cdot \pi_i = \cdots = \frac{28}{81},$$

so that the realized offered traffic becomes

$$a^r_{\text{offered}} = \lambda^h / \mu = \frac{28}{27} = 1.037.$$ 

This can be expressed in an equivalent form as follows:

$$a^r_{\text{offered}} = \frac{k \nu}{\nu (1 - B_c) + \mu} = \frac{k(\frac{1}{2})}{1 + (\frac{\nu}{\mu})(1 - B_c)} = \frac{4(\frac{1}{4})}{1 + (\frac{1}{3})(1 - \frac{1}{7})} = \frac{28}{27} = 1.037.$$ 

Finally, the carried traffic satisfies

$$a_{\text{carried}} = a^r_{\text{offered}} (1 - B_c) = \frac{28}{27}(1 - \frac{1}{7}) = \frac{8}{9} = 0.89$$

D7/3 (a) Figure 1.

(b) We see from Figure 1 that the Markov process $X(t)$ is irreducible (L6/10). Since the state space is finite, the equilibrium distribution $\pi$ exists, and it can be derived based on the global balance equations (GBE) and the normalization condition (N), cf. L6/11.

Let us start with the GBE’s for states 0 and 2:

$$\pi_0 \lambda = \pi_1 \mu, \quad \pi_2 2 \mu = \pi_0 \frac{\lambda}{2} + \pi_1 \lambda$$
The other probabilities are now solved as a function of $\pi_0$:

$$
\pi_1 = \pi_0 \frac{\lambda}{\mu}, \quad \pi_2 = \pi_0 \left( \frac{1}{4} \frac{\lambda}{\mu} + \frac{1}{2} \frac{\lambda^2}{\mu^2} \right)
$$

The remaining probability $\pi_0$ is determined by (N):

$$
\pi_0 + \pi_1 + \pi_2 = \pi_0 \left( 1 + \frac{5}{4} \frac{\lambda}{\mu} + \frac{1}{2} \frac{\lambda^2}{\mu^2} \right) = 1.
$$

Thus, the equilibrium distribution is

$$
\pi_0 = \frac{1}{1 + \frac{5}{4} \frac{\lambda}{\mu} + \frac{1}{2} \frac{\lambda^2}{\mu^2}}, \quad \pi_1 = \frac{\lambda}{1 + \frac{5}{4} \frac{\lambda}{\mu} + \frac{1}{2} \frac{\lambda^2}{\mu^2}}, \quad \pi_2 = \frac{\frac{1}{4} \frac{\lambda}{\mu} + \frac{1}{2} \frac{\lambda^2}{\mu^2}}{1 + \frac{5}{4} \frac{\lambda}{\mu} + \frac{1}{2} \frac{\lambda^2}{\mu^2}}
$$

If $\lambda = \mu$ (as will be assumed in (c) and (d)), the equilibrium distribution is

$$
\pi_0 = \frac{4}{11} = 0.36, \quad \pi_1 = \frac{4}{11} = 0.36, \quad \pi_2 = \frac{3}{11} = 0.27
$$

(c) The mean number of busy servers is

$$
E[X_S] = \sum_{i=0}^{n} i \cdot \pi_i = \frac{4}{11} + 2 \cdot \frac{3}{11} = \frac{10}{11} = 0.91
$$

Thus, the utilization factor becomes

$$
E[U] = \frac{E[X_S]}{n} = \frac{\frac{10}{11}}{2} = \frac{5}{11} = 0.45
$$

(d) To calculate the call blocking probability, we need to know the mean number of customers in a batch, $E[A]$, and the mean number of lost customers in a batch, $E[L]$. The former one is clearly

$$
E[A] = 1 \cdot P\{A = 1\} + 2 \cdot P\{A = 2\} = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{3}{2} = 1.50
$$

On the other hand, due to the PASTA property of the Poisson process (L5/28), an arriving batch sees the system in equilibrium. Thus,

$$
E[L] = \pi_1 (1 \cdot P\{A = 2\}) + \pi_2 (1 \cdot P\{A = 1\} + 2 \cdot P\{A = 2\})
$$

$$
= \pi_1 P\{A = 2\} + \pi_2 E[A] = \frac{4}{11} \cdot \frac{1}{2} + \frac{3}{11} \cdot \frac{3}{2} = \frac{13}{22} = 0.59
$$
The call blocking probability $B_c$ is their ratio:

$$B_c = \frac{E[L]}{E[A]} = \frac{\frac{13}{2}}{\frac{3}{2}} = \frac{13}{3} = 0.39$$

**Note:** The traffic intensity is now

$$a = \frac{\lambda E[A]}{\mu} = E[A] = \frac{3}{2} = 1.50$$

If the customers arrived individually (and not in batches) according to a Poisson process, the blocking probability would be, by the Erlang formula (L7/20),

$$\text{Erl}(n, a) = \frac{a^n}{\sum_{j=0}^{n} \frac{n!}{j!}} = \frac{\left(\frac{3}{2}\right)^2}{1 + \frac{3}{2} + \frac{3}{2}^2} = \frac{9}{8 + 12 + 9} = \frac{9}{29} = 0.31,$$

which is less than for the original system. So a burstier arrival process results in a higher blocking probability.