TKK HELSINKI UNIVERSITY OF TECHNOLOGY — **Demonstrations**
Department of Communications and Networking — **Lecture 6**
S-38.1145 Introduction to Teletraffic Theory, Spring 2008 — 1.2.2008

D6/1 A Markov process is defined in the state space $\{0, 1, 2\}$ with the state transitions rates q_{ij} collected in the transition matrix $Q = (q_{ij} \mid i, j = 0, 1, 2)$, where $q_{ii} = -q_i$ for all i, as follows:

$$Q = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & \mu & -(1+\mu) \end{pmatrix}$$

(a) Draw the state transition diagram of $X(t)$.

(b) Derive the equilibrium distribution of $X(t)$.

(c) Describe the behaviour of the system in equilibrium when μ is very large or very small, respectively.

D6/2 A Markov process is defined in the state space $\{0, 1, 2, 3\}$ with the state transitions rates q_{ij} collected in the transition matrix $Q = (q_{ij} \mid i, j = 0, 1, 2, 3)$, where $q_{ii} = -q_i$ for all i, as follows:

$$Q = \begin{pmatrix} -3 & 2 & 0 & 1 \\ 1 & -3 & 2 & 0 \\ 0 & 1 & -3 & 2 \\ 2 & 0 & 1 & -3 \end{pmatrix}$$

(a) Draw the state transition diagram of $X(t)$.

(b) Derive the equilibrium distribution of $X(t)$.

(c) Is the process reversible? (In other words, are the local balance equations (LBE) satisfied?)

D6/3 The Ehrenfest model was used, at a time, to shed light on a paradox related to the second law of thermodynamics. The model can be described as follows. A closed system is constructed from N randomly moving gas molecules and two containers which are connected so that each of the molecules changes the container at rate λ, independently of the other molecules. Let $X(t)$ denote the number of molecules in one of the containers, which clearly is a Markov process.

(a) Draw the state transition diagram of $X(t)$.

(b) Derive the equilibrium distribution of $X(t)$.

c) Compare the conditional probabilities $P\{X(t) = N \mid X(0) = N/2\}$ and $P\{X(t) = N/2 \mid X(0) = N\}$ when N is even and t is very large.

D6/1 (a) Figure 1, cf. L6/12.

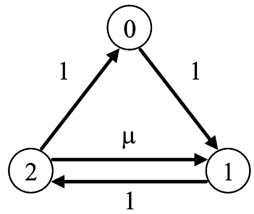

Figure 1: [D6/1] State transition diagram.

(b) We see from Figure 1 that the Markov process $X(t)$ is irreducible (L6/10). Since the state space is finite, the equilibrium distribution π exists, and it can be derived based on the global balance equations (GBE) and the normalization condition (N), cf. L6/11.

Let us start with the GBE's for states 0 and 1:

$$\pi_0 = \pi_2, \quad \pi_1 = \pi_0 + \pi_2\mu$$

The other probabilities are now solved as a function of π_0:

$$\pi_1 = \pi_0(1+\mu), \quad \pi_2 = \pi_0$$

The remaining probability π_0 is determined by (N):

$$\pi_0 + \pi_1 + \pi_2 = \pi_0(1 + (1+\mu) + 1) = \pi_0(3+\mu) = 1.$$

Thus, the equilibrium distribution is

$$\pi_0 = \frac{1}{3+\mu}, \quad \pi_1 = \frac{1+\mu}{3+\mu}, \quad \pi_2 = \frac{1}{3+\mu}$$

(c) If $\mu \to \infty$, then the process is almost always at state 1 ($\pi_1 \to 1$ and $\pi_i \to 0$ for the other i).

But if $\mu \to 0$, then the process ultimately visits the states in the order $0 \to 1 \to 2 \to 0$ staying in each state on average 1 time unit. This results in the uniform equilibrium distribution ($\pi_i \to 1/3$ for all i).

D6/2 (a) Figure 2.

(b) We see from Figure 2 that the Markov process $X(t)$ is irreducible (L6/10). Since the state space is finite, the equilibrium distribution π exists. The figure also tells us that all the states are symmetric (with respect to the state transition rates), which implies that the equilibrium distribution be the uniform distribution with probibilities

$$\pi_0 = \pi_1 = \pi_2 = \pi_3 = \frac{1}{4}$$

Just in case, you can easily check that both the global balance equations (GBE)

$$3\pi_0 = 2\pi_3 + \pi_1, \quad 3\pi_1 = 2\pi_0 + \pi_2, \quad 3\pi_2 = 2\pi_1 + \pi_3, \quad 3\pi_3 = 2\pi_2 + \pi_0$$

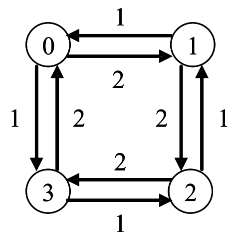

Figure 2: [D6/2] State transition diagram.

and the normalization condition (N)

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

are satisfied for this uniform distribution.

(c) Consider, for example, the local balance equation between states 0 and 1:

$$2\pi_0 = \pi_1$$

The uniform equilibrium distribution ($\pi_0 = \pi_1 = 1/4$) does *not* satisfy this condition. Thus, the process is not reversible.

D6/3 (a) Figure 3.

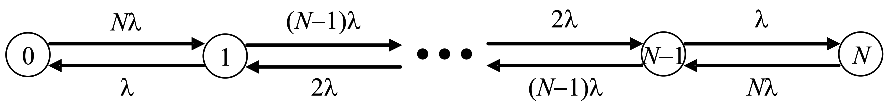

Figure 3: [D6/3] State transition diagram.

(b) We see from Figure 3 that $X(t)$ is an irreducible birth-death-process (L6/16). Since the state space is finite, the equilibrium distribution π exists, and it can be derived based on the local balance equations (LBE) and the normalization condition (N), cf. L6/17.
Let us start with the LBE's for states $i-1$ and i, where $i = 1, \ldots, N$:

$$\pi_{i-1}(N-i+1)\lambda = \pi_i \, i\lambda$$

This results in the following recursion:

$$\begin{aligned}
\pi_i &= \pi_{i-1}\,\frac{N-i+1}{i} \\
&= \pi_{i-2}\,\frac{(N-i+1)(N-i+2)}{i(i-1)} = \ldots \\
&= \pi_0\,\frac{(N-i+1)(N-i+2)\cdots N}{i(i-1)\cdots 1} \\
&= \pi_0\,\frac{N!}{i!(N-i)!} = \pi_0 \begin{pmatrix} N \\ i \end{pmatrix}
\end{aligned}$$

The remaining probability π_0 is determined by (N):

$$\pi_0 + \pi_1 + \ldots + \pi_N = \pi_0 \sum_{i=0}^{N} \binom{N}{i} = \pi_0\, 2^N = 1,$$

Thus,

$$\pi_0 = \left(\frac{1}{2}\right)^N,$$

and the equilibrium distribution is the binomial distribution $\mathrm{Bin}(N, 1/2)$,

$$\pi_i = \binom{N}{i} \left(\frac{1}{2}\right)^N, \qquad i = 0, 1, \ldots, N$$

(c) As $t \to \infty$, the effect of the initial state disappears and the state probabilities approach the equilibrium distribution. Thus,

$$P\{X(t) = N \mid X(0) = N/2\} \to \pi_N \quad \text{and} \quad P\{X(t) = N/2 \mid X(0) = N\} \to \pi_{N/2}$$

For example, for $N = 50$,

$$\pi_N = 8.9 \cdot 10^{-16} \quad \text{and} \quad \pi_{N/2} = 0.11.$$

So the probability of a state in the middle (of the state transition diagram) is much greater than at the edge. The entropy of the system tends to increase, and the molecules spread evenly to both containers.