D4/1 Let \( X \) and \( Y \) be two independent discrete random variables. Define \( Z = aX + bY \), where \( a, b \neq 0 \).

(a) Determine the expectation \( E[Z] \) and the variance \( D^2[Z] \).

(b) Calculate the expectation \( E[Z] \) and the probability \( P\{Z = 0\} \) assuming that \( X \sim \text{Poisson}(3.0), Y \sim \text{Poisson}(1.75) \), and \( a = b = 10 \).

(b) Calculate the expectation \( E[Z] \) and the probability \( P\{Z = 0\} \) assuming that \( X \sim \text{Poisson}(3.0), Y \sim \text{Poisson}(1.75) \), and \( a = b = 100 \).

D4/2 Let \( X \) and \( Y \) be two discrete random variables such that \( Y = \min\{X, n\} \), where \( n \) is a given constant. Define \( Z = aX + bY \), where \( a, b \neq 0 \).

(a) Determine the expectation \( E[Z] \).

(b) Calculate the expectation \( E[Z] \) and the probability \( P\{Z = 0\} \) assuming that \( X \sim \text{Poisson}(3.0), a = b = 10, \) and \( n = 2 \).

D4/3 Let \( X \sim \text{Exp}(\lambda) \), and assume that \( P\{X \leq 10.0\} = 0.8 \).

(a) Determine the parameter \( \lambda \).

(b) Determine the expectation \( E[X] \).

(c) Determine \( t \) such that \( P\{X \leq t\} = 1/2 \) (which is called median).

(d) Determine the coefficient of variation \( C[X] \).

D4/4 Let \( X \sim \text{Pareto}(\beta, b) \) with value set \( S_X = (0, \infty) \) and cumulative distribution function

\[
F_X(x) = 1 - \left( \frac{1}{1 + bx} \right)^\beta, \quad x > 0.
\]

The parameter \( \beta \) is called the shape parameter, and the parameter \( b \) is the scale parameter.

(a) Determine the expectation \( E[X] \) and the variance \( D^2[X] \).

(b) Determine the coefficient of variation \( C[X] \). Make a comparison with the corresponding parameter of the exponential distribution \( \text{Exp}(\lambda) \).
D4/1 (a) Since expectation is a linear operator (L4/21), we have
\[ E[Z] = E[aX + bY] \overset{(i)}{=} E[aX] + E[bY] \overset{(i)}{=} aE[X] + bE[Y] \]

If \( X \) and \( Y \) are independent (L4/20), then \( aX \) and \( bY \) are independent, too, since for any \( x \) and \( y \) we have
\[ P\{aX = x, bY = y\} = P\{X = x/a, Y = y/b\} = P\{X = x\}\{Y = y\} = P\{aX = x\}\{bY = y\} \]

Thus, the variance takes the following form (L4/22):
\[ D^2[Z] = D^2[aX + bY] \overset{(ii)}{=} D^2[aX] + D^2[bY] \overset{(i)}{=} a^2D^2[X] + b^2D^2[Y] \]

(b) Assume that \( a = b = 10 \). Since \( E[X] = 3.0 \) and \( E[Y] = 1.75 \), we have
\[ E[Z] \overset{(a)}{=} aE[X] + bE[Y] = 10 \cdot 3.0 + 10 \cdot 1.75 = 47.5 \]

In addition, since \( X \) and \( Y \) are nonnegative and \( a, b > 0 \), \( Z = aX + bY = 0 \) if and only if \( X = 0 \) and \( Y = 0 \). Therefore
\[ P\{Z = 0\} \overset{L4/20}{=} P\{X = 0, Y = 0\} \overset{L4/33}{=} e^{-3.0}e^{-1.75} = e^{-4.75} \approx 0.009 \]

(c) Assume now that \( a = b = 100 \). It follows that the expectation becomes tenfold,
\[ E[Z] \overset{(a)}{=} aE[X] + bE[Y] = 100 \cdot 3.0 + 100 \cdot 1.75 = 475.0, \]

but the probability \( P\{Z = 0\} \) remains the same:
\[ P\{Z = 0\} = P\{X = 0, Y = 0\} \overset{(b)}{=} e^{-4.75} \approx 0.009 \]

D4/2 (a) As in the previous problem, we have (L4/21)
\[ E[Z] = E[aX + bY] \overset{(i)}{=} E[aX] + E[bY] \overset{(i)}{=} aE[X] + bE[Y] \]

Note that it is not required here that the variables be independent (which they are not in this case).

(b) Assume that \( a = b = 10 \) and \( n = 2 \). Now \( E[X] = 3.0 \) and
\[ E[Y] \overset{L4/21}{=} E[\min\{X, 2\}] \]
\[ = P\{X = 1\} + 2P\{X \geq 2\} = P\{X = 1\} + 2(1 - P\{X < 2\}) \overset{L4/33}{=} 3.0 \cdot e^{-3.0} + 2 \cdot (1 - e^{-3.0} - 3.0 \cdot e^{-3.0}) = 2 - 5 \cdot e^{-3.0} \approx 1.75 \]
Thus,
\[ E[Z] = aE[X] + bE[Y] = 10 \cdot 3 + 10 \cdot 1.75 = 47.5 \]

Again, as in the previous problem, \( Z = aX + bY = 0 \) if and only if \( X = 0 \) and \( Y = 0 \). On the other hand, \( Y = \min\{X,n\} = 0 \) if and only if \( X = 0 \). It follows that

\[
P\{Z = 0\} = P\{X = 0, Y = 0\} = P\{X = 0\} \quad \text{L4/33} \quad e^{-3.0} \quad \text{num.} \quad 0.050
\]

**D4/3**

(a) Since

\[
P\{X \leq 10.0\} \quad \text{L4/42} \quad 1 - e^{-10.0\lambda} = 0.8,
\]

we have

\[
\lambda = -\frac{1}{10.0} \log(1 - 0.8) = -\frac{1}{10.0} \log(0.2) \quad \text{num.} \quad 0.16
\]

(b) Mean value:

\[
E[X] \quad \text{L4/42} \quad \frac{1}{\lambda} = 6.21
\]

(c) Since

\[
P\{X \leq t\} \quad \text{L4/42} \quad 1 - e^{-\lambda t} = \frac{1}{2},
\]

we have

\[
t = -\frac{1}{\lambda} \log\left(1 - \frac{1}{2}\right) = 10.0 \cdot \frac{\log(0.5)}{\log(0.2)} \quad \text{num.} \quad 4.31
\]

(d) For any exponential distribution, the coefficient of variation is the same (independent of the parameter \( \lambda \)):

\[
C[X] = \frac{D[X]}{E[X]} = \frac{\sqrt{D^2[X]}}{E[X]} = \frac{L4/42}{L4/24} \quad \frac{1}{\lambda} = 1.
\]

**D4/4**

(a) Let us first determine the density function for the Pareto distribution:

\[
f_X(x) = \frac{L4/39}{F_X'(x)} = b\beta \left(\frac{1}{1 + bx}\right)^{\beta+1}.
\]

Mean value:

\[
E[X] = \int_0^\infty x f_X(x) \, dx = \frac{\int_0^\infty x \cdot b\beta \left(\frac{1}{1 + bx}\right)^{\beta+1} \, dx}{\text{os.int.}} = \int_0^\infty \left(\frac{1}{1 + bx}\right)^\beta \, dx = \frac{1}{b(\beta - 1)}
\]

The mean value is finite only for \( \beta > 1 \).
Second moment:

\[
E[X^2] = \int_0^\infty x^2 f_X(x) \, dx = \int_0^\infty x^2 b^\beta \left( \frac{1}{1 + bx} \right)^{\beta+1} \, dx
\]

\[
= \frac{2}{b(\beta - 1)} \int_0^\infty \left( \frac{1}{1 + bx} \right)^{\beta-1} \, dx = \frac{2}{b^2(\beta - 1)(\beta - 2)}
\]

The second moment is finite only for \( \beta > 2 \).

Variance:

\[
D^2[X] = E[X^2] - E[X]^2 = \frac{2}{b^2(\beta - 1)(\beta - 2)} - \frac{1}{b^2(\beta - 1)^2} = \frac{\beta}{b^2(\beta - 1)^2(\beta - 2)}
\]

The variance is finite only for \( \beta > 2 \).

(b) Coefficient of variation (for \( \beta > 2 \)):

\[
C[X] = \frac{D[X]}{E[X]} = \sqrt{\frac{D^2[X]}{E[X]}} = \sqrt{\frac{\beta}{\beta - 2}} > 1
\]

Thus, for the Pareto(\( \beta, b \)) distribution, the coefficient of variation depends only on the shape parameter \( \beta \). In addition, it is always greater than the coefficient of variation 1 of any exponential distribution.