**D4/1** Let $X$ and $Y$ be two independent discrete random variables. Define $Z = aX + bY$, where $a, b \neq 0$.

(a) Determine the expectation $E[Z]$ and the variance $D^2[Z]$.

(b) Calculate the expectation $E[Z]$ and the probability $P\{Z = 0\}$ assuming that $X \sim \text{Poisson}(3.0)$, $Y \sim \text{Poisson}(1.75)$, and $a = b = 10$.

(b) Calculate the expectation $E[Z]$ and the probability $P\{Z = 0\}$ assuming that $X \sim \text{Poisson}(3.0)$, $Y \sim \text{Poisson}(1.75)$, and $a = b = 100$.

**D4/2** Let $X$ and $Y$ be two discrete random variables such that $Y = \min\{X, n\}$, where $n$ is a given constant. Define $Z = aX + bY$, where $a, b \neq 0$.

(a) Determine the expectation $E[Z]$.

(b) Calculate the expectation $E[Z]$ and the probability $P\{Z = 0\}$ assuming that $X \sim \text{Poisson}(3.0)$, $a = b = 10$, and $n = 2$.

**D4/3** Let $X \sim \text{Exp}(\lambda)$, and assume that $P\{X \leq 10.0\} = 0.8$.

(a) Determine the parameter $\lambda$.

(b) Determine the expectation $E[X]$.

(c) Determine $t$ such that $P\{X \leq t\} = 1/2$ (which is called median).

(d) Determine the coefficient of variation $C[X]$.

**D4/4** Let $X \sim \text{Pareto}(\beta, b)$ with value set $S_X = (0, \infty)$ and cumulative distribution function

$$F_X(x) = 1 - \left(\frac{1}{1 + bx}\right)^\beta, \qquad x > 0.$$

The parameter $\beta$ is called the shape parameter, and the parameter $b$ is the scale parameter.

(a) Determine the expectation $E[X]$ and the variance $D^2[X]$.

(b) Determine the coefficient of variation $C[X]$. Make a comparison with the corresponding parameter of the exponential distribution $\text{Exp}(\lambda)$.

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**D4/1** (a) Since expectation is a linear operator (L4/21), we have

$$E[Z] = E[aX + bY] \overset{\text{(ii)}}{=} E[aX] + E[bY] \overset{\text{(i)}}{=} aE[X] + bE[Y]$$

If $X$ and $Y$ are inependent (L4/20), then $aX$ and $bY$ are independent, too, since for any $x$ and $y$ we have

$$\begin{aligned} P\{aX = x, bY = y\} &= P\{X = x/a, Y = y/b\} \\ &= P\{X = x/a\}\{Y = y/b\} \\ &= P\{aX = x\}\{bY = y\} \end{aligned}$$

Thus, the variance takes the following form (L4/22):

$$D^2[Z] = D^2[aX + bY] \overset{\text{(ii)}}{=} D^2[aX] + D^2[bY] \overset{\text{(i)}}{=} a^2D^2[X] + b^2D^2[Y]$$

(b) Assume that $a = b = 10$. Since $E[X] = 3.0$ and $E[Y] = 1.75$, we have

$$E[Z] \overset{\text{(a)}}{=} aE[X] + bE[Y] = 10 \cdot 3.0 + 10 \cdot 1.75 = 47.5$$

In addition, since $X$ and $Y$ are nonnegative and $a, b > 0$, $Z = aX + bY = 0$ if and only if $X = 0$ and $Y = 0$. Therefore

$$\begin{aligned} P\{Z = 0\} &= P\{X = 0, Y = 0\} \\ &\overset{\text{L4/20}}{=} P\{X = 0\}P\{Y = 0\} \\ &\overset{\text{L4/33}}{=} e^{-3.0}e^{-1.75} = e^{-4.75} \overset{\text{num.}}{=} 0.009 \end{aligned}$$

(c) Assume now that $a = b = 100$. It follows that the expectation becomes tenfold,

$$E[Z] \overset{\text{(a)}}{=} aE[X] + bE[Y] = 100 \cdot 3.0 + 100 \cdot 1.75 = 475.0,$$

but the probability $P\{Z = 0\}$ remains the same:

$$P\{Z = 0\} = P\{X = 0, Y = 0\} \overset{\text{(b)}}{=} e^{-4.75} \overset{\text{num.}}{=} 0.009$$

**D4/2** (a) As in the previous problem, we have (L4/21)

$$E[Z] = E[aX + bY] \overset{\text{(ii)}}{=} E[aX] + E[bY] \overset{\text{(i)}}{=} aE[X] + bE[Y]$$

Note that it is not required here that the variables be independent (which they are not in this case).

(b) Assume that $a = b = 10$ and $n = 2$. Now $E[X] = 3.0$ and

$$\begin{aligned} E[Y] &= E[\min\{X, 2\}] \\ &\overset{\text{L4/21}}{=} P\{X = 1\} + 2P\{X \geq 2\} = P\{X = 1\} + 2(1 - P\{X < 2\}) \\ &\overset{\text{L4/33}}{=} 3.0 \cdot e^{-3.0} + 2 \cdot (1 - e^{-3.0} - 3.0 \cdot e^{-3.0}) = 2 - 5 \cdot e^{-3.0} \overset{\text{num.}}{=} 1.75 \end{aligned}$$

Thus,

$$E[Z] = aE[X] + bE[Y] = 10 \cdot 3 + 10 \cdot 1.75 = 47.5$$

Again, as in the previous problem, $Z = aX + bY = 0$ if and only if $X = 0$ and $Y = 0$. On the other hand, $Y = \min\{X, n\} = 0$ if and only if $X = 0$. It follows that

$$P\{Z = 0\} = P\{X = 0, Y = 0\} = P\{X = 0\} \overset{\text{L4/33}}{=} e^{-3.0} \overset{\text{num.}}{=} 0.050$$

**D4/3** (a) Since

$$P\{X \leq 10.0\} \overset{\text{L4/42}}{=} 1 - e^{-10.0 \cdot \lambda} = 0.8,$$

we have

$$\lambda = -\frac{1}{10.0} \log(1 - 0.8) = -\frac{1}{10.0} \log(0.2) \overset{\text{num.}}{=} 0.16$$

(b) Mean value:

$$E[X] \overset{\text{L4/42}}{=} \frac{1}{\lambda} = 6.21$$

(c) Since

$$P\{X \leq t\} \overset{\text{L4/42}}{=} 1 - e^{-\lambda t} = \frac{1}{2},$$

we have

$$t = -\frac{1}{\lambda} \log(1 - \frac{1}{2}) = 10.0 \cdot \frac{\log(0.5)}{\log(0.2)} \overset{\text{num.}}{=} 4.31$$

(d) For any exponential distribution, the coefficient of variation is the same (independent of the parameter $\lambda$):

$$C[X] \overset{\text{L4/24}}{=} \frac{D[X]}{E[X]} \overset{\text{L4/24}}{=} \frac{\sqrt{D^2[X]}}{E[X]} \overset{\text{L4/42}}{=} \frac{1/\lambda}{1/\lambda} = 1.$$

**D4/4** (a) Let us first determine the density function for the Pareto distribution:

$$f_X(x) \overset{\text{L4/37}}{=} F'_X(x) = b\beta \left( \frac{1}{1 + bx} \right)^{\beta + 1}.$$

Mean value:

$$\begin{aligned} E[X] &\overset{\text{L4/39}}{=} \int_0^\infty x\, f_X(x)\, dx \\ &= \int_0^\infty x\, b\beta \left( \frac{1}{1 + bx} \right)^{\beta+1} dx \\ &\overset{\text{os.int.}}{=} \int_0^\infty \left( \frac{1}{1 + bx} \right)^{\beta} dx = \frac{1}{b(\beta - 1)} \end{aligned}$$

The mean value is finite only for $\beta > 1$.

Second moment:

$$\begin{aligned} E[X^2] &\overset{\text{L4/39}}{=} \int_0^\infty x^2 \, f_X(x) \, dx \\ &= \int_0^\infty x^2 \, b\beta \left( \frac{1}{1+bx} \right)^{\beta+1} dx \\ &\overset{\text{os.int.}}{=} \frac{2}{b(\beta-1)} \int_0^\infty \left( \frac{1}{1+bx} \right)^{\beta-1} dx = \frac{2}{b^2(\beta-1)(\beta-2)} \end{aligned}$$

The second moment is finite only for $\beta > 2$.
Variance:

$$\begin{aligned} D^2[X] &\overset{\text{L4/22}}{=} E[X^2] - E[X]^2 \\ &= \frac{2}{b^2(\beta-1)(\beta-2)} - \frac{1}{b^2(\beta-1)^2} = \frac{\beta}{b^2(\beta-1)^2(\beta-2)} \end{aligned}$$

The variance is finite only for $\beta > 2$.

(b) Coefficient of variation (for $\beta > 2$):

$$C[X] \overset{\text{L4/24}}{=} \frac{D[X]}{E[X]} \overset{\text{L4/24}}{=} \frac{\sqrt{D^2[X]}}{E[X]} \overset{\text{(a)}}{=} \sqrt{\frac{\beta}{\beta-2}} > 1$$

Thus, for the Pareto$(\beta, b)$ distribution, the coefficient of variation depends only on the shape parameter $\beta$. In addition, it is always greater than the coefficient of variation 1 of any exponential distribution.