4. Basic probability theory

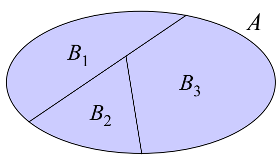
Contents

Sample space, sample points, events

- **Sample space** Ω is the set of all possible **sample points** $\omega \in \Omega$
  - **Example 0**. Tossing a coin: $\Omega = \{H,T\}$
  - **Example 1**. Casting a die: $\Omega = \{1,2,3,4,5,6\}$
  - **Example 2**. Number of customers in a queue: $\Omega = \{0,1,2,...\}$
  - **Example 3**. Call holding time (e.g. in minutes): $\Omega = \{x \in \Re \mid x > 0\}$
- **Events** $A,B,C,... \subset \Omega$ are measurable subsets of the sample space Ω
  - **Example 1**. "Even numbers of a die": $A = \{2,4,6\}$
  - **Example 2**. "No customers in a queue": $A = \{0\}$
  - **Example 3**. "Call holding time greater than 3.0 (min)": $A = \{x \in \Re \mid x > 3.0\}$
- Denote by $\mathfrak{I}$ the set of all events $A \in \mathfrak{I}$
  - **Sure event**: The sample space $\Omega \in \mathfrak{I}$ itself
  - **Impossible event**: The empty set $\emptyset \in \mathfrak{I}$

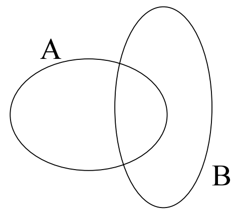
Combination of events

- **Union** "A or B": $A \cup B = \{\omega \in \Omega \mid \omega \in A \text{ or } \omega \in B\}$
- **Intersection** "A and B": $A \cap B = \{\omega \in \Omega \mid \omega \in A \text{ and } \omega \in B\}$
- **Complement** "not A": $A^c = \{\omega \in \Omega \mid \omega \notin A\}$
- Events A and B are **disjoint** if
  - $A \cap B = \emptyset$
- A set of events $\{B_1, B_2, \ldots\}$ is a **partition** of event A if
  - (i) $B_i \cap B_j = \emptyset$ for all $i \neq j$
  - (ii) $\cup_i B_i = A$
  - **Example 1**. Odd and even numbers of a die constitute a partition of the sample space: $B_1 = \{1,3,5\}$ and $B_2 = \{2,4,6\}$



Probability

- **Probability** of event A is denoted by $P(A)$, $P(A) \in [0,1]$
  - Probability measure P is thus
    a real-valued set function defined on the set of events $\mathcal{F}$, $P: \mathcal{F} \to [0,1]$
- **Properties**:
  - (i) $0 \le P(A) \le 1$
  - (ii) $P(\emptyset) = 0$
  - (iii) $P(\Omega) = 1$
  - (iv) $P(A^c) = 1 - P(A)$
  - (v) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - (vi) $A \cap B = \emptyset \Rightarrow \quad P(A \cup B) = P(A) + P(B)$
  - (vii) $\{B_i\}$ is a partition of $A \Rightarrow \; P(A) = \Sigma_i P(B_i)$
  - $(viii)$ $A \subset B \Rightarrow P(A) \le P(B)$



Conditional probability

- Assume that $P(B) > 0$
- **Definition**: The **conditional probability** of event A **given** that event B occurred is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- It follows that

$$P(A \cap B) = P(B)P(A \mid B) = P(A)P(B \mid A)$$

Theorem of total probability

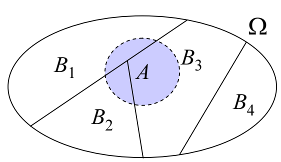
- Let $\{B_i\}$ be a partition of the sample space Ω
- It follows that $\{A \cap B_i\}$ is a partition of event A. Thus (by slide 5)

$$P(A) \overset{(vii)}{=} \textstyle\sum_i P(A \cap B_i)$$

- Assume further that $P(B_i) > 0$ for all i. Then (by slide 6)

$$P(A) = \textstyle\sum_i P(B_i) P(A \mid B_i)$$

- This is the **theorem of total probability**



Bayes' theorem

- Let $\{B_i\}$ be a partition of the sample space Ω
- Assume that $P(A) > 0$ and $P(B_i) > 0$ for all i. Then (by slide 6)

$$P(B_i \mid A) = \frac{P(A \cap B_i)}{P(A)} = \frac{P(B_i)P(A \mid B_i)}{P(A)}$$

- Furthermore, by the theorem of total probability (slide 7), we get

$$P(B_i \mid A) = \frac{P(B_i)P(A \mid B_i)}{\sum_j P(B_j)P(A \mid B_j)}$$

- This is **Bayes' theorem**
  - Probabilities $P(B_i)$ are called ***a priori*** probabilities of events B_i
  - Probabilities $P(B_i \mid A)$ are called ***a posteriori*** probabilities of events B_i (given that the event A occured)

Statistical independence of events

- **Definition**: Events A and B are **independent** if

$$P(A \cap B) = P(A)P(B)$$

- It follows that

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

- Correspondingly:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B)$$

Random variables

- **Definition**: Real-valued **random variable** X is a real-valued and measurable function defined on the sample space Ω, $X\colon \Omega \to \Re$
  - Each sample point $\omega \in \Omega$ is associated with a real number $X(\omega)$
- **Measurability** means that all sets of type

$$\{X \le x\} := \{\omega \in \Omega \mid X(\omega) \le x\} \subset \Omega$$

belong to the set of events $\mathcal{F}$, that is

$$\{X \le x\} \in \mathcal{F}$$

- The probability of such an event is denoted by $P\{X \le x\}$

Example

- A coin is tossed three times
- Sample space:

$$\Omega = \{(\omega_1, \omega_2, \omega_3) \mid \omega_i \in \{\mathrm{H}, \mathrm{T}\}, i = 1,2,3\}$$

- Let X be the random variable that tells the total number of tails in these three experiments:

ω	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
$X(\omega)$	0	1	1	1	2	2	2	3

Indicators of events

- Let $A \in \mathcal{F}$ be an arbitrary event
- **Definition**: The **indicator** of event A is a random variable defined as follows:

$$1_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

- Clearly:

$$P\{1_A = 1\} = P(A)$$

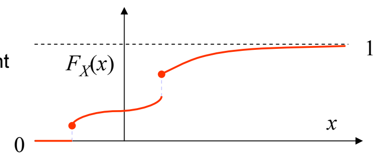
$$P\{1_A = 0\} = P(A^c) = 1 - P(A)$$

Cumulative distribution function

- **Definition**: The **cumulative distribution function** (cdf) of a random variable X is a function $F_X: \Re \to [0,1]$ defined as follows:

$$F_X(x) = P\{X \le x\}$$

- Cdf determines the **distribution** of the random variable,
  - that is: the probabilities $P\{X \in B\}$, where $B \subset \Re$ and $\{X \in B\} \in \mathcal{F}$
- **Properties**:
  - (i) F_X is non-decreasing
  - (ii) F_X is continuous from the right
  - (iii) $F_X(-\infty) = 0$
  - (iv) $F_X(\infty) = 1$



Statistical independence of random variables

- **Definition**: Random variables X and Y are **independent** if for all x and y

$$P\{X \le x, Y \le y\} = P\{X \le x\}P\{Y \le y\}$$

- **Definition**: Random variables $X_1,\ldots,X_n$ are **totally independent** if for all i and x_i

$$P\{X_1 \le x_1,\ldots,X_n \le x_n\} = P\{X_1 \le x_1\}\cdots P\{X_n \le x_n\}$$

Maximum and minimum of independent random variables

- Let the random variables $X_1,\ldots,X_n$ be **totally independent**
- Denote: $X^{\max} := \max\{X_1,\ldots,X_n\}$. Then

$$P\{X^{\max} \le x\} = P\{X_1 \le x,\ldots,X_n \le x\}$$
$$= P\{X_1 \le x\}\cdots P\{X_n \le x\}$$

- Denote: $X^{\min} := \min\{X_1,\ldots,X_n\}$. Then

$$P\{X^{\min} > x\} = P\{X_1 > x,\ldots,X_n > x\}$$
$$= P\{X_1 > x\}\cdots P\{X_n > x\}$$

Contents

Discrete random variables

- **Definition**: Set $A \subset \Re$ is called **discrete** if it is
  - finite, $A = \{x_1, \ldots, x_n\}$, or
  - countably infinite, $A = \{x_1, x_2, \ldots\}$
- **Definition**: Random variable X is **discrete** if there is a discrete set $S_X \subset \Re$ such that

$$P\{X \in S_X\} = 1$$

- It follows that
  - $P\{X = x\} \geq 0$ for all $x \in S_X$
  - $P\{X = x\} = 0$ for all $x \notin S_X$
- The set S_X is called the **value set**

Point probabilities

- Let X be a discrete random variable
- The distribution of X is determined by the **point probabilities** p_i,

$$p_i := P\{X = x_i\}, \quad x_i \in S_X$$

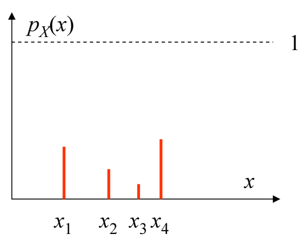
- **Definition**: The **probability mass function** (pmf) of X is a function $p_X : \Re \to [0,1]$ defined as follows:

$$p_X(x) := P\{X = x\} = \begin{cases} p_i, & x = x_i \in S_X \\ 0, & x \notin S_X \end{cases}$$

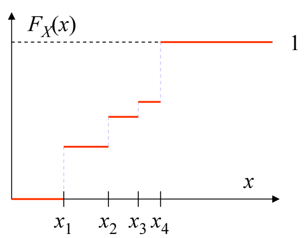
- Cdf is in this case a step function:

$$F_X(x) = P\{X \leq x\} = \sum_{i: x_i \leq x} p_i$$

Example



probability mass function (pmf)



cumulative distribution function (cdf)

$$S_X = \{x_1, x_2, x_3, x_4\}$$

Independence of discrete random variables

- Discrete random variables X and Y are independent if and only if for all $x_i \in S_X$ and $y_j \in S_Y$

$$P\{X = x_i, Y = y_j\} = P\{X = x_i\} P\{Y = y_j\}$$

Expectation

- **Definition**: The **expectation** (mean value) of X is defined by

$$\mu_X := E[X] := \sum_{x \in S_X} P\{X = x\} \cdot x = \sum_{x \in S_X} p_X(x) x = \sum_i p_i x_i$$

  - Note 1: The expectation exists only if $\Sigma_i p_i |x_i| < \infty$
  - Note 2: If $\Sigma_i p_i x_i = \infty$, then we may denote $E[X] = \infty$

- **Properties**:
  - (i) $c \in \Re \Rightarrow E[cX] = cE[X]$
  - (ii) $E[X + Y] = E[X] + E[Y]$
  - (iii) X and Y independent $\Rightarrow E[XY] = E[X]E[Y]$

Variance

- **Definition**: The **variance** of X is defined by

$$\sigma_X^2 := D^2[X] := \mathrm{Var}[X] := E[(X - E[X])^2]$$

- Useful formula (prove!):

$$D^2[X] = E[X^2] - E[X]^2$$

- **Properties**:
  - (i) $c \in \Re \Rightarrow D^2[cX] = c^2 D^2[X]$
  - (ii) X and Y independent $\Rightarrow D^2[X + Y] = D^2[X] + D^2[Y]$

Covariance

- **Definition**: The **covariance** between X and Y is defined by

$$\sigma_{XY}^2 := \mathrm{Cov}[X,Y] := E[(X - E[X])(Y - E[Y])]$$

- Useful formula (prove!):

$$\mathrm{Cov}[X,Y] = E[XY] - E[X]E[Y]$$

- **Properties**:
  - (i) $\mathrm{Cov}[X,X] = \mathrm{Var}[X]$
  - (ii) $\mathrm{Cov}[X,Y] = \mathrm{Cov}[Y,X]$
  - (iii) $\mathrm{Cov}[X+Y,Z] = \mathrm{Cov}[X,Z] + \mathrm{Cov}[Y,Z]$
  - (iv) X and Y independent $\Rightarrow \mathrm{Cov}[X,Y] = 0$

Other distribution related parameters

- **Definition**: The **standard deviation** of X is defined by

$$\sigma_X := D[X] := \sqrt{D^2[X]}$$

- **Definition**: The **coefficient of variation** of X is defined by

$$c_X := C[X] := \frac{D[X]}{E[X]}$$

- **Definition**: The kth **moment,** $k=1,2,\ldots,$ of X is defined by

$$\mu_X^{(k)} := E[X^k]$$

Average of IID random variables

- Let $X_1,\ldots,X_n$ be independent and identically distributed (**IID**) with mean μ and variance σ^2
- Denote the average (sample mean) as follows:

$$\bar{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i$$

- Then (prove!)

$$E[\bar{X}_n] = \mu$$

$$D^2[\bar{X}_n] = \frac{\sigma^2}{n}$$

$$D[\bar{X}_n] = \frac{\sigma}{\sqrt{n}}$$

Law of large numbers (LLN)

- Let $X_1,\ldots,X_n$ be independent and identically distributed (**IID**) with mean μ and variance σ^2
- **Weak law of large numbers**: for all $\varepsilon > 0$

$$P\{|\bar{X}_n - \mu| > \varepsilon\} \to 0$$

- **Strong law of large numbers**: with probability 1

$$\bar{X}_n \to \mu$$

- It follows that for large values of n

$$\bar{X}_n \approx \mu$$

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Bernoulli distribution

$$X \sim \text{Bernoulli}(p), \quad p \in (0,1)$$

  - describes a simple random experiment with two possible outcomes: success (1) and failure (0); cf. coin tossing
  - success with probability p (and failure with probability $1-p$)
- Value set: $S_X = \{0,1\}$
- Point probabilities:

$$P\{X=0\} = 1-p, \quad P\{X=1\} = p$$

- Mean value: $E[X] = (1-p)\cdot 0 + p\cdot 1 = p$
- Second moment: $E[X^2] = (1-p)\cdot 0^2 + p\cdot 1^2 = p$
- Variance: $D^2[X] = E[X^2] - E[X]^2 = p - p^2 = p(1-p)$

Binomial distribution

$$X \sim \mathrm{Bin}(n, p), \quad n \in \{1,2,\ldots\}, p \in (0,1)$$

- number of successes in an independent series of simple random experiments (of Bernoulli type); $X = X_1 + \ldots + X_n$ (with $X_i \sim \mathrm{Bernoulli}(p)$)
- n = total number of experiments
- p = probability of success in any single experiment

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

$$n! = n \cdot (n-1) \cdots 2 \cdot 1$$

- Value set: $S_X = \{0,1,\ldots,n\}$
- Point probabilities:

$$P\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i}$$

- Mean value: $E[X] = E[X_1] + \ldots + E[X_n] = np$
- Variance: $D^2[X] = D^2[X_1] + \ldots + D^2[X_n] = np(1-p)$ (independence!)

Geometric distribution

$$X \sim \mathrm{Geom}(p), \quad p \in (0,1)$$

- number of successes until the first failure in an independent series of simple random experiments (of Bernoulli type)
- p = probability of success in any single experiment

- Value set: $S_X = \{0,1,\ldots\}$
- Point probabilities:

$$P\{X = i\} = p^i (1-p)$$

- Mean value: $E[X] = \sum_i ip^i(1-p) = p/(1-p)$
- Second moment: $E[X^2] = \sum_i i^2 p^i(1-p) = 2(p/(1-p))^2 + p/(1-p)$
- Variance: $D^2[X] = E[X^2] - E[X]^2 = p/(1-p)^2$

Memoryless property of geometric distribution

- Geometric distribution has so called **memoryless property**: for all $i,j \in \{0,1,\ldots\}$

$$P\{X \geq i + j \mid X \geq i\} = P\{X \geq j\}$$

- Prove!
  - *Tip*: Prove first that $P\{X \geq i\} = p^i$

Minimum of geometric random variables

- Let $X_1 \sim \mathrm{Geom}(p_1)$ and $X_2 \sim \mathrm{Geom}(p_2)$ be **independent**. Then

$$X^{\min} := \min\{X_1, X_2\} \sim \mathrm{Geom}(p_1 p_2)$$

and

$$P\{X^{\min} = X_i\} = \frac{1-p_i}{1-p_1 p_2}, \quad i \in \{1,2\}$$

- Prove!
  - *Tip*: See slide 15

Poisson distribution

$$X \sim \text{Poisson}(a), \quad a > 0$$

- limit of binomial distribution as $n \to \infty$ and $p \to 0$ in such a way that $np \to a$

- Value set: $S_X = \{0,1,\ldots\}$
- Point probabilities:

$$P\{X = i\} = \frac{a^i}{i!} e^{-a}$$

- Mean value: $E[X] = a$
- Second moment: $E[X(X-1)] = a^2 \Rightarrow E[X^2] = a^2 + a$
- Variance: $D^2[X] = E[X^2] - E[X]^2 = a$

Example

- Assume that
  - 200 subscribers are connected to a local exchange
  - each subscriber's characteristic traffic is 0.01 erlang
  - subscribers behave independently
- Then the number of active calls $X \sim \text{Bin}(200,0.01)$
- Corresponding Poisson-approximation $X \approx \text{Poisson}(2.0)$
- Point probabilities:

	0	1	2	3	4	5
Bin(200,0.01)	.1326	.2679	.2693	.1795	.0893	.0354
Poisson(2.0)	.1353	.2701	.2701	.1804	.0902	.0361

Properties

- (i) **Sum**: Let $X_1 \sim \text{Poisson}(a_1)$ and $X_2 \sim \text{Poisson}(a_2)$ be independent. Then

$$X_1 + X_2 \sim \text{Poisson}(a_1 + a_2)$$

- (ii) **Random sample**: Let $X \sim \text{Poisson}(a)$ denote the number of elements in a set, and Y denote the size of a random sample of this set (each element taken independently with probability p). Then

$$Y \sim \text{Poisson}(pa)$$

- (iii) **Random sorting**: Let X and Y be as in (ii), and $Z = X - Y$. Then Y and Z are **independent** (given that X is unknown) and

$$Z \sim \text{Poisson}((1-p)a)$$

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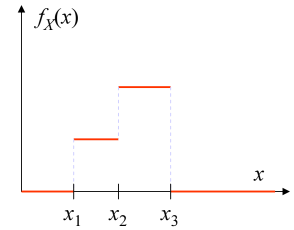
Continuous random variables

- **Definition**: Random variable X is **continuous** if there is an integrable function f_X: $\Re \to \Re_+$ such that for all $x \in \Re$

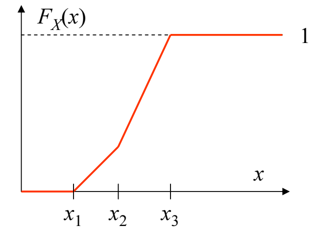
$$F_X(x) := P\{X \le x\} = \int_{-\infty}^{x} f_X(y)\,dy$$

- The function f_X is called the **probability density function** (pdf)
  - The set S_X, where $f_X > 0$, is called the **value set**
- Properties:
  - (i) $P\{X = x\} = 0$ for all $x \in \Re$
  - (ii) $P\{a < X < b\} = P\{a \le X \le b\} = \int_a^b f_X(x)\,dx$
  - (iii) $P\{X \in A\} = \int_A f_X(x)\,dx$
  - (iv) $P\{X \in \Re\} = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_{S_X} f_X(x)\,dx = 1$

Example



probability density function (pdf)



cumulative distribution function (cdf)

$$S_X = [x_1, x_3]$$

Expectation and other distribution related parameters

- **Definition**: The **expectation** (mean value) of X is defined by

$$\mu_X := E[X] := \int_{-\infty}^{\infty} f_X(x)\,x\,dx$$

  - Note 1: The expectation exists only if $\int_{-\infty}^{\infty} f_X(x)|x|\,dx < \infty$
  - Note 2: If $\int_{-\infty}^{\infty} f_X(x)x = \infty$, then we may denote $E[X] = \infty$
  - The expectation has the same properties as in the discrete case (see slide 21)
- The other distribution parameters (variance, covariance,...) are defined just as in the discrete case
  - These parameters have the same properties as in the discrete case (see slides 22-24)

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Uniform distribution

$$X \sim \mathrm{U}(a,b), \quad a < b$$

- continuous counterpart of "casting a die"

- Value set: $S_X = (a,b)$
- Probability density function (pdf):

$$f_X(x) = \frac{1}{b-a}, \quad x \in (a,b)$$

- Cumulative distribution function (cdf):

$$F_X(x) := P\{X \le x\} = \frac{x-a}{b-a}, \quad x \in (a,b)$$

- Mean value: $E[X] = \int_a^b x/(b-a)\,dx = (a+b)/2$
- Second moment: $E[X^2] = \int_a^b x^2/(b-a)\,dx = (a^2+ab+b^2)/3$
- Variance: $D^2[X] = E[X^2] - E[X]^2 = (b-a)^2/12$

Exponential distribution

$$X \sim \mathrm{Exp}(\lambda), \quad \lambda > 0$$

- continuous counterpart of geometric distribution ("failure" prob. $\approx \lambda dt$)
- $P\{X \in (t,t+h] \mid X > t\} = \lambda h + o(h)$, where $o(h)/h \to 0$ as $h \to 0$

- Value set: $S_X = (0,\infty)$
- Probability density function (pdf):

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0$$

- Cumulative distribution function (cdf):

$$F_X(x) = P\{X \le x\} = 1 - e^{-\lambda x}, \quad x > 0$$

- Mean value: $E[X] = \int_0^\infty \lambda x \exp(-\lambda x)\,dx = 1/\lambda$
- Second moment: $E[X^2] = \int_0^\infty \lambda x^2 \exp(-\lambda x)\,dx = 2/\lambda^2$
- Variance: $D^2[X] = E[X^2] - E[X]^2 = 1/\lambda^2$

Memoryless property of exponential distribution

- Exponential distribution has so called **memoryless property**: for all $x,y \in (0,\infty)$

$$P\{X > x+y \mid X > x\} = P\{X > y\}$$

- Prove!
  - *Tip*: Prove first that $P\{X > x\} = e^{-\lambda x}$
- Application:
  - Assume that the call holding time is exponentially distributed with mean h minutes.
  - Consider a call that has already lasted for x minutes. Due to memoryless property, this gives **no information about** the length of the **remaining holding time**: it is distributed as the original holding time and, on average, lasts still h minutes!

Minimum of exponential random variables

- Let $X_1 \sim \mathrm{Exp}(\lambda_1)$ and $X_2 \sim \mathrm{Exp}(\lambda_2)$ be **independent**. Then

$$X^{\min} := \min\{X_1, X_2\} \sim \mathrm{Exp}(\lambda_1 + \lambda_2)$$

and

$$P\{X^{\min} = X_i\} = \frac{\lambda_i}{\lambda_1 + \lambda_2}, \quad i \in \{1,2\}$$

- Prove!
  - *Tip*: See slide 15

Standard normal (Gaussian) distribution

$$X \sim \mathrm{N}(0,1)$$

- limit of the "normalized" sum of IID r.v.s with mean 0 and variance 1 (cf. slide 48)

- Value set: $S_X = (-\infty, \infty)$
- Probability density function (pdf):

$$f_X(x) = \varphi(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

- Cumulative distribution function (cdf):

$$F_X(x) := P\{X \le x\} = \Phi(x) := \int_{-\infty}^{x} \varphi(y)\, dy$$

- Mean value: $E[X] = 0$ (symmetric pdf)
- Variance: $D^2[X] = 1$

Normal (Gaussian) distribution

$$X \sim \mathrm{N}(\mu, \sigma^2), \quad \mu \in \Re, \quad \sigma > 0$$

- if $(X - \mu)/\sigma \sim \mathrm{N}(0,1)$

- Value set: $S_X = (-\infty, \infty)$
- Probability density function (pdf):

$$f_X(x) = F_X'(x) = \tfrac{1}{\sigma} \varphi\left(\tfrac{x-\mu}{\sigma}\right)$$

- Cumulative distribution function (cdf):

$$F_X(x) := P\{X \le x\} = P\left\{\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right\} = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

- Mean value: $E[X] = \mu + \sigma E[(X - \mu)/\sigma] = \mu$ (symmetric pdf around μ)
- Variance: $D^2[X] = \sigma^2 D^2[(X - \mu)/\sigma] = \sigma^2$

Properties of the normal distribution

- (i) **Linear transformation**: Let $X \sim \mathrm{N}(\mu, \sigma^2)$ and $\alpha, \beta \in \Re$. Then

$$Y := \alpha X + \beta \sim \mathrm{N}(\alpha\mu + \beta, \alpha^2\sigma^2)$$

- (ii) **Sum**: Let $X_1 \sim \mathrm{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathrm{N}(\mu_2, \sigma_2^2)$ be **independent**. Then

$$X_1 + X_2 \sim \mathrm{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

- (iii) **Sample mean**: Let $X_i \sim \mathrm{N}(\mu, \sigma^2)$, $i = 1, \ldots n$, be independent and identically distributed (**IID**). Then (cf. slide 25)

$$\bar{X}_n := \tfrac{1}{n} \sum_{i=1}^{n} X_i \sim \mathrm{N}(\mu, \tfrac{1}{n}\sigma^2)$$

Central limit theorem (CLT)

- Let $X_1, \ldots, X_n$ be **independent and identically distributed** (**IID**) with mean μ and variance σ^2 (and the third moment exists)
- **Central limit theorem**:

$$\frac{1}{\sigma/\sqrt{n}} (\bar{X}_n - \mu) \xrightarrow{\text{i.d.}} \mathrm{N}(0,1)$$

- It follows that for large values of n

$$\bar{X}_n \approx \mathrm{N}(\mu, \tfrac{1}{n}\sigma^2)$$

Contents

Other random variables

- In addition to discrete and continuous random variables, there are so called **mixed** random variables
  - containing some discrete as well as continuous portions
- Example:
  - The customer waiting time W in an M/M/1 queue has an **atom** at zero ($P\{W = 0\} = 1 - \rho > 0$) but otherwise the distribution is continuous